# A proof system for the modal $\mu$-calculus inspired by the determinisation of automata

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# Outline

- Present non-wellfounded proof systems for the modal $\mu$-calculus
- Show connections to automata theory
- Introduce determinisation method for parity automata
- Define proof system using this method
- Discuss benefits of this system

# Modal $\mu$-calculus

The *formulas* in the modal $\mu$-calculus are generated by the grammar

$$\varphi \;::=\; p \mid \overline{p} \mid \bot \mid \top \mid (\varphi \vee \varphi) \mid (\varphi \wedge \varphi) \mid \Diamond\varphi \mid \Box\varphi \mid \mu x\,\varphi \mid \nu x\,\varphi$$

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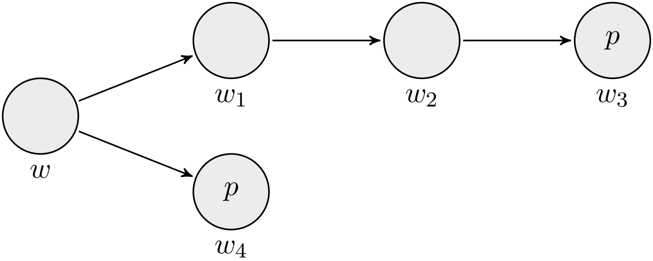
- Formulas of the form $\mu x \ \varphi$ and $\nu x \ \varphi$ are called *fixpoint formulas* and interpreted as the least and greatest fixpoint of $\varphi$
- In $\mu x \ \varphi$ and $\nu x \ \varphi$ there are no occurrences of $\overline{x}$ in $\varphi$
- A fixpoint formula $\varphi$ is *more important* than a fixpoint formula $\psi$ if $\psi$ is a subformula of $\varphi$

# Example

Let $\mathcal{M} = (W, R, V)$ be the following Kripke model



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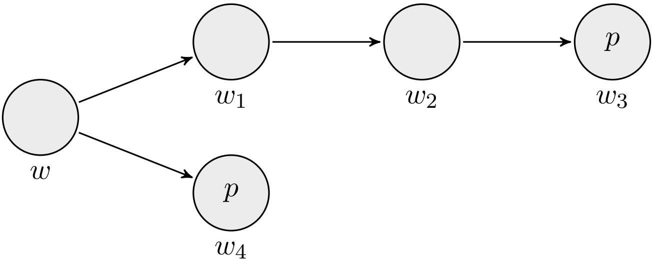


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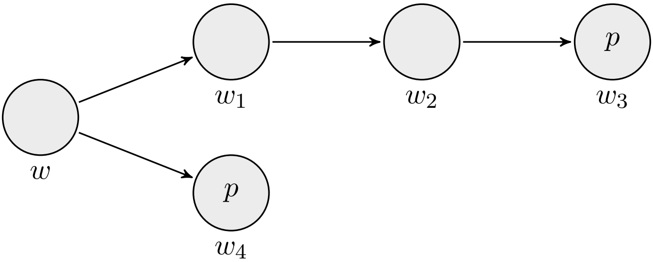


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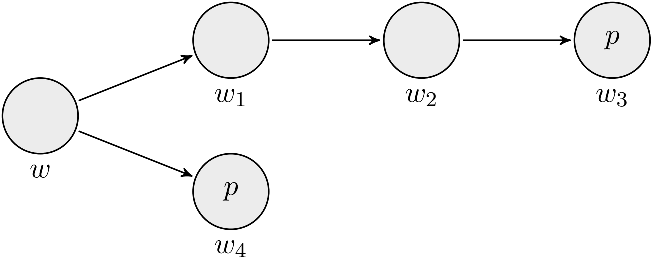


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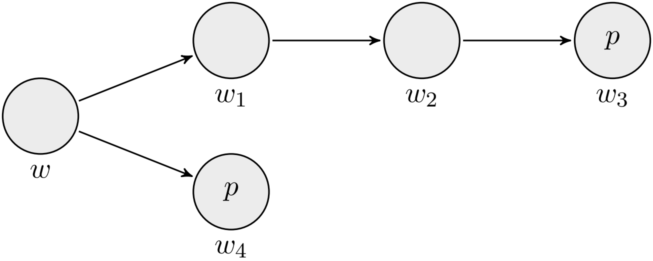


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# Proof theory of the modal $\mu$-calculus

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# NW pre-proofs

An NW *pre-proof* is a, possibly infinite, tree defined from the following rules:

Ax1: $\dfrac{}{p, \bar{p}, \Gamma}$ Ax2: $\dfrac{}{\top, \Gamma}$ R$_\vee$: $\dfrac{\varphi, \psi, \Gamma}{\varphi \vee \psi, \Gamma}$ R$_\wedge$: $\dfrac{\varphi, \Gamma \qquad \psi, \Gamma}{\varphi \wedge \psi, \Gamma}$

R$_\Box$: $\dfrac{\varphi, \Gamma}{\Box\varphi, \Diamond\Gamma, \Delta}$ R$_\mu$: $\dfrac{\varphi[\mu x.\varphi/x], \Gamma}{\mu x.\varphi, \Gamma}$ R$_\nu$: $\dfrac{\varphi[\nu x.\varphi/x], \Gamma}{\nu x.\varphi, \Gamma}$

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- There are infinite branches
- But only finitely many sequents

# Example

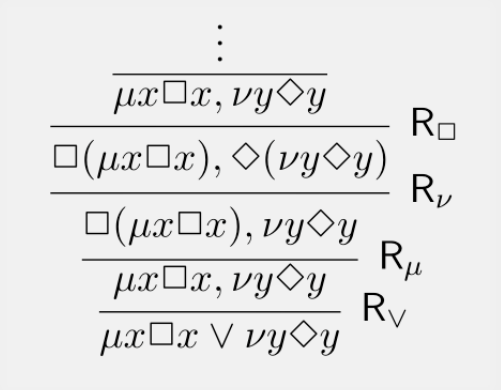


Figure: NW pre-proof of $\mu x \Box x \vee \nu y \Diamond y$

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- A *trace* $(\varphi_j)_{j\in\omega}$ on an infinite branch is an infinite sequence of formulas such that $\varphi_j$ is an immediate ancestor of $\varphi_{j+1}$ for $j \in \omega$.

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**Definition**

An NW proof is an NW pre-proof, where on every infinite branch there is a $\nu$-trace.

# Example

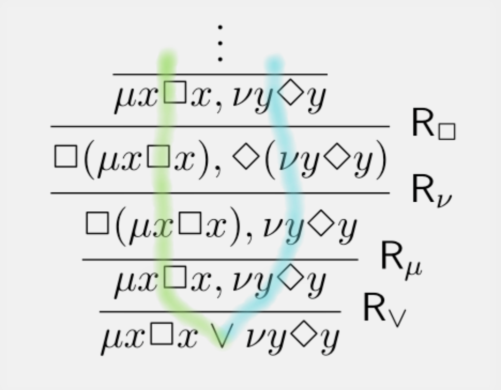


Figure: NW proof of $\mu x \Box x \lor \nu y \Diamond y$

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Variation of finite state automaton which has infinite strings as inputs

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## Definition

Let $\Sigma$ be a finite set, called an *alphabet*. A *non-deterministic automaton* over $\Sigma$ is a quadruple $\mathbb{A} = \langle A, \Delta, a_I, \mathrm{Acc}\rangle$, where $A$ is a finite set, $\Delta : A \times \Sigma \to \mathcal{P}(A)$ is the transition function of $\mathbb{A}$, $a_I \in A$ its initial state and $\mathrm{Acc} \subseteq A^\omega$ its acceptance condition.

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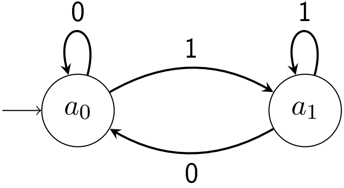
Let $\Sigma = \{0, 1\}$ and $\mathbb{A} = \langle A, \Delta, a_I, \mathrm{Acc} \rangle$ be given as



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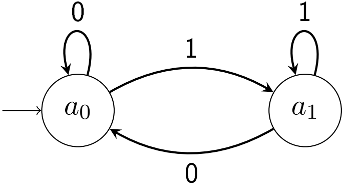


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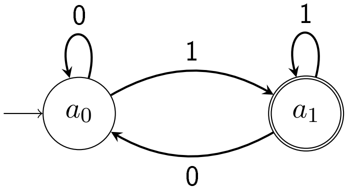


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- A *Büchi* condition is given as a subset $F \subseteq A$. The corresponding acceptance condition is the set of runs, which contain infinitely many states in $F$.

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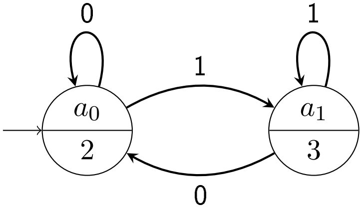


The acceptance condition can be given in different ways:

- A *parity* condition is given as a map $\Omega : A \to \omega$. The corresponding acceptance condition is the set of runs $\alpha$ such that $\max\{\Omega(a) \mid a \text{ occurs infinitely often in } \alpha\}$ is even.

# Tracking automaton

We can define nondeterministic parity automaton $\mathbb{A}$ s.t. for all infinite branches $\alpha$ in an NW pre-proof:

$$\mathbb{A} \text{ accepts } \alpha \Leftrightarrow \text{ there is a } \nu\text{-trace on } \alpha$$

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Idea:

- States are formulas
- Transitions given by ancestor relation
- Parity of fixpoint formulas:
  - $\nu$-formulas get even parity
  - $\mu$-formulas get odd parity
  - More important fixpoint formulas get higher parity

# Obtaining new proof system

Idea: build automaton into proof system

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Need automaton to be deterministic!

Let $\mathbb{A}^D$ be deterministic automaton accepting same language as $\mathbb{A}$

- Sequents of form $a \vdash \Gamma$, where $a$ state of $\mathbb{A}^D$

Main advantage: Soundness condition based on branches instead of traces

# Explicit determinisation

- Most known determinisation method is Safra construction
- Inspired by it [Jungteerapanich '10] and [Stirling '14] introduced annotated proof system
    - Sequents have form $\theta \vdash \varphi_1^{\rho_1}, ..., \varphi_n^{\rho_n}$

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- States of deterministic automaton $\mathbb{B}$ consists of
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- We develop determinisation method for nondeterministic automata using binary trees
- States of deterministic automaton $\mathbb{B}$ consists of
  - Sets of states of $\mathbb{A}$
  - Every state is annotated by tuple of binary strings
- Using this method we get a different annotated proof system
  - Sequents have form $\vdash \varphi_1^{\sigma_1}, ..., \varphi_n^{\sigma_n}$
  - No extra information needed!

# BT proof rules

Ax1: $$\frac{}{p^\sigma, \bar{p}^\tau, \Gamma}$$ Ax2: $$\frac{}{\top^\sigma, \Gamma}$$ R$_\vee$: $$\frac{\varphi^\sigma, \psi^\sigma, \Gamma}{(\varphi \vee \psi)^\sigma, \Gamma}$$ R$_\wedge$: $$\frac{\varphi^\sigma, \Gamma \quad \psi^\sigma, \Gamma}{(\varphi \wedge \psi)^\sigma, \Gamma}$$

R$_\Box$: $$\frac{\varphi^\sigma, \Gamma}{\Box\varphi^\sigma, \Diamond\Gamma, \Delta}$$ R$_\nu$: $$\frac{\varphi[x\backslash\nu x.\varphi]^{\sigma\restriction k\cdot 1_k}, \Gamma^{\cdot 0_k}}{\nu x.\varphi^\sigma, \Gamma}$$ where $k = \Omega_\Phi(\nu x.\varphi)$

R$_\mu$: $$\frac{\varphi[x\backslash\mu x.\varphi]^{\sigma\restriction\Omega_\Phi(\mu x.\varphi)}, \Gamma}{\mu x.\varphi^\sigma, \Gamma}$$ Resolve: $$\frac{\varphi^\sigma, \Gamma}{\varphi^\sigma, \varphi^\tau, \Gamma}$$ where $\sigma > \tau$

Compress$_k^{s0}$: $$\frac{\varphi_1^{(\ldots,st_1,\ldots)}, \ldots, \varphi_n^{(\ldots,st_n,\ldots)}, \Gamma}{\varphi_1^{(\ldots,s0t_1,\ldots)}, \ldots, \varphi_n^{(\ldots,s0t_n,\ldots)}, \Gamma}$$ where $s \notin \Gamma_k^A$

Compress$_k^{s1}$: $$\frac{\varphi_1^{(\ldots,st_1,\ldots)}, \ldots, \varphi_n^{(\ldots,st_n,\ldots)}, \Gamma}{\varphi_1^{(\ldots,s1t_1,\ldots)}, \ldots, \varphi_n^{(\ldots,s1t_n,\ldots)}, \Gamma}$$ where $s \notin \Gamma_k^A$ and $s \neq 0 \cdots 0$

# BT$^\infty$ proofs

**Definition**

A BT$^\infty$ proof is a BT pre-proof, where on every infinite branch there is a successful string.

- Completeness and Soundness of BT$^\infty$ proven by using determinisation method
- Advantage: Soundness condition on branches instead of traces

# Example proof

$$
\begin{array}{cl}
\vdots & \\
\hline
\mu x \Box x^{0}, \nu y \Diamond y^{1} & \\
\hline
\Box(\mu x \Box x)^{0}, \Diamond(\nu y \Diamond y)^{1} & \mathsf{R}_{\Box} \\
\hline
\Box(\mu x \Box x)^{0}, \Diamond(\nu y \Diamond y)^{11} & \mathsf{Compress}^{11} \\
\hline
\Box(\mu x \Box x)^{0}, \nu y \Diamond y^{1} & \mathsf{R}_{\nu} \\
\hline
\mu x \Box x^{0}, \nu y \Diamond y^{1} & \mathsf{R}_{\mu} \\
\hline
\Box(\mu x \Box x)^{0}, \Diamond(\nu y \Diamond y)^{1} & \mathsf{R}_{\Box} \\
\hline
\Box(\mu x \Box x)^{\epsilon}, \nu y \Diamond y^{\epsilon} & \mathsf{R}_{\nu} \\
\hline
\mu x \Box x^{\epsilon}, \nu y \Diamond y^{\epsilon} & \mathsf{R}_{\mu} \\
\hline
\mu x \Box x \vee \nu y \Diamond y^{\epsilon} & \mathsf{R}_{\vee}
\end{array}
$$

# BT proofs

- Only finitely many sequents on infinite branch
- Add discharge rule:

$$\mathsf{D}^{\mathsf{x}}: \frac{\begin{array}{c}[\Gamma]^{\mathsf{x}} \\ \vdots \\ \Gamma\end{array}}{\Gamma}$$

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- Get cyclic proof tree
- Infinite branches correspond to strongly connected components

**Definition**

A BT proof is a finite BT pre-proof, where for every strongly connected subgraph there is a successful string.

- Comparing to Jungteerapanich system: Trade-off between extra information and stronger soundness condition

# Example proof

$$
\begin{array}{cl}
[\mu x\Box x^0, \nu y\Diamond y^1]^{\mathsf{x}} & \\ \hline
\Box(\mu x\Box x)^0, \Diamond(\nu y\Diamond y)^1 & \mathsf{R}_\Box \\ \hline
\Box(\mu x\Box x)^0, \Diamond(\nu y\Diamond y)^{11} & \mathsf{Compress}^{11} \\ \hline
\Box(\mu x\Box x)^0, \nu y\Diamond y^1 & \mathsf{R}_\nu \\ \hline
\mu x\Box x^0, \nu y\Diamond y^1 & \mathsf{R}_\mu \\ \hline
\mu x\Box x^0, \nu y\Diamond y^1 & \mathsf{D}^{\mathsf{x}} \\ \hline
\Box(\mu x\Box x)^0, \Diamond(\nu y\Diamond y)^1 & \mathsf{R}_\Box \\ \hline
\Box(\mu x\Box x)^\epsilon, \nu y\Diamond y^\epsilon & \mathsf{R}_\nu \\ \hline
\mu x\Box x^\epsilon, \nu y\Diamond y^\epsilon & \mathsf{R}_\mu \\ \hline
\mu x\Box x \vee \nu y\Diamond y^\epsilon & \mathsf{R}_\vee
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$$

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- Further work:
    - Compare to Jungteerapanich proof system
    - Translate BT proofs to proofs of Kozen's finitary proof system

Thank you !