A proof system for the modal μ-calculus inspired by the determinisation of automata

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Outline

- Present different proof systems for the modal μ-calculus
- Show connections to automata theory
- Explain benefits of our system

Modal μ-calculus

The *formulas* in the modal μ-calculus are generated by the grammar

$$\varphi \ ::= \ p \mid \overline{p} \mid \bot \mid \top \mid (\varphi \vee \varphi) \mid (\varphi \wedge \varphi) \mid \Diamond\varphi \mid \Box\varphi \mid \mu x\, \varphi \mid \nu x\, \varphi$$

- p and x are taken from a fixed set of propositional variables,
- Formulas of the form $\mu x\ \varphi$ and $\nu x\ \varphi$ are called *fixpoint formulas* and interpreted as the least and greatest fixpoint of φ,
- In $\mu x\ \varphi$ and $\nu x\ \varphi$ there are no occurrences of $\overline{x}$ in φ.

Proof systems for the modal μ-calculus

- [Kozen '83] introduced finitary proof system with explicit induction rules
- Completeness proven by [Walukiewicz '00]
- [Niwiński, Walukiewicz '96] introduced infinitary tableaux games in which one player has winning strategy iff formula is valid

NW proofs

An NW *pre-proof* is a, possibly infinite, tree defined from the following rules:

Ax1: $\dfrac{}{p, \bar{p}, \Gamma}$ Ax2: $\dfrac{}{\top, \Gamma}$ $\mathsf{R}_\vee$: $\dfrac{\varphi, \psi, \Gamma}{\varphi \vee \psi, \Gamma}$ $\mathsf{R}_\wedge$: $\dfrac{\varphi, \Gamma \qquad \psi, \Gamma}{\varphi \wedge \psi, \Gamma}$

$\mathsf{R}_\Box$: $\dfrac{\varphi, \Gamma}{\Box\varphi, \Diamond\Gamma, \Delta}$ R_μ: $\dfrac{\varphi[\mu x.\varphi/x], \Gamma}{\mu x.\varphi, \Gamma}$ R_ν: $\dfrac{\varphi[\nu x.\varphi/x], \Gamma}{\nu x.\varphi, \Gamma}$

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R$_\Box$: $\dfrac{\varphi, \Gamma}{\Box\varphi, \Diamond\Gamma, \Delta}$ R$_\mu$: $\dfrac{\varphi[\mu x.\varphi/x], \Gamma}{\mu x.\varphi, \Gamma}$ R$_\nu$: $\dfrac{\varphi[\nu x.\varphi/x], \Gamma}{\nu x.\varphi, \Gamma}$

- A *trace* $(\varphi_j)_{j\in\omega}$ on an infinite branch is an infinite sequence of formulas such that φ_{j+1} is an ancestor of φ_j for $j \in \omega$.
- A trace is called ν-*trace* if the most important fixpoint formula unfolded infinitely often is a ν-formula.

Definition

An NW proof is an NW pre-proof, where on every infinite branch there is a ν-trace.

Automata

- The trace condition can be checked by a nondeterministic ω-automaton
- Idea: build automaton into system
    - Sequents of form $a \vdash \Gamma$, where a state of automaton
- Need automaton to be deterministic
- [Jungteerapanich '10] and [Stirling '14] introduced annotated proof system inspired by the Safra construction
    - Sequents have form $\theta \vdash \varphi_1^{\rho_1}, ..., \varphi_n^{\rho_n}$
- This system has been reduced to Kozen's axiomatization by [Afshari, Leigh '17]

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- We develop determinisation method for nondeterministic automata using binary trees
- Using this method we get a different annotated proof system
  - Sequents have form $\vdash \varphi_1^{\sigma_1}, ..., \varphi_n^{\sigma_n}$
  - No extra information needed!

BT proof rules

Ax1: $\dfrac{}{p^\sigma, \bar{p}^\tau, \Gamma}$ Ax2: $\dfrac{}{\top^\sigma, \Gamma}$ R$_\vee$: $\dfrac{\varphi^\sigma, \psi^\sigma, \Gamma}{(\varphi \vee \psi)^\sigma, \Gamma}$ R$_\wedge$: $\dfrac{\varphi^\sigma, \Gamma \quad \psi^\sigma, \Gamma}{(\varphi \wedge \psi)^\sigma, \Gamma}$

R$_\Box$: $\dfrac{\varphi^\sigma, \Gamma}{\Box\varphi^\sigma, \Diamond\Gamma, \Delta}$ R$_\nu$: $\dfrac{\varphi[x \backslash \nu x.\varphi]^{\sigma \restriction k \cdot 1_k}, \Gamma^{\cdot 0_k}}{\nu x.\varphi^\sigma, \Gamma}$ where $k = \Omega_\Phi(\nu x.\varphi)$

R$_\mu$: $\dfrac{\varphi[x \backslash \mu x.\varphi]^{\sigma \restriction \Omega_\Phi(\mu x.\varphi)}, \Gamma}{\mu x.\varphi^\sigma, \Gamma}$ Resolve: $\dfrac{\varphi^\sigma, \Gamma}{\varphi^\sigma, \varphi^\tau, \Gamma}$ where $\sigma > \tau$

Compress$_k^{s0}$: $\dfrac{\varphi_1^{(\ldots, st_1, \ldots)}, \ldots, \varphi_n^{(\ldots, st_n, \ldots)}, \Gamma}{\varphi_1^{(\ldots, s0t_1, \ldots)}, \ldots, \varphi_n^{(\ldots, s0t_n, \ldots)}, \Gamma}$ where $s \notin \Gamma_k^A$

Compress$_k^{s1}$: $\dfrac{\varphi_1^{(\ldots, st_1, \ldots)}, \ldots, \varphi_n^{(\ldots, st_n, \ldots)}, \Gamma}{\varphi_1^{(\ldots, s1t_1, \ldots)}, \ldots, \varphi_n^{(\ldots, s1t_n, \ldots)}, \Gamma}$ where $s \notin \Gamma_k^A$ and $s \neq 0 \cdots 0$

BT proofs

Definition

A BT^{∞} proof is a BT pre-proof, where on every infinite branch there is a successful string.

Definition

A BT proof is the cyclic version of a BT^{∞} proof, where for every strongly connected subgraph there is a successful string.

- BT^{∞} and BT are sound and complete proof systems for the μ-calculus
- BT proofs can be translated to Clo proofs introduced by Afshari and Leigh
- This yields completeness proof of Kozen's axiomatization

Future work

- Check complexity of translations
- Compare to Jungteerapanich proof system
- Generalize method, to make it applicable for other cyclic proof systems:
    - Cyclic Arithmetic
    - $CLKID^{\omega}$ for first-order logic with inductive definitions
    - ...

Thank you !