Interpolation with cyclic proofs

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Outline

- What is interpolation?
- How to prove it using cyclic proofs
- Case studies:
  - Two-way modal μ-calculus
  - Converse PDL

Interpolation

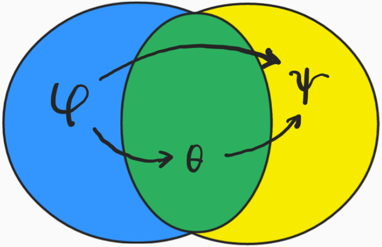
Interpolation

Definition

A logic has Craig interpolation if for any two formulas φ and ψ such that $\varphi \to \psi$ is valid, there is an interpolant θ with

- $\varphi \to \theta$ and $\theta \to \psi$ valid
- $\mathsf{Voc}(\theta) \subseteq \mathsf{Voc}(\varphi) \cap \mathsf{Voc}(\psi)$

$\mathsf{Voc}(\varphi)$: set of proposition letters and modalities in φ



Interpolation

Applications:

- model checking
- knowledge representation
- ...

Proof methods:

- model theory
- algebra
- proof theory

Interpolation

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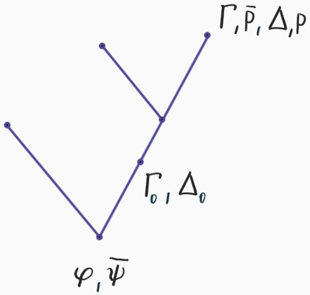
Proof methods:

- model theory
- algebra
- proof theory: Maehara's method

Maehara's method

Setup:

- start with (cut-free) proof system
  - $\hookrightarrow$ read *conjunctively*:
    - $\vdash \Gamma$ iff $\bigwedge \Gamma$ is *unsatisfiable*
  - $\hookrightarrow \vdash \varphi, \overline{\psi}$ iff $\varphi \to \psi$ is valid



Maehara's method

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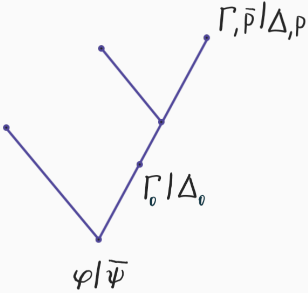
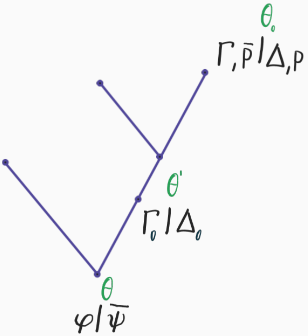
  $\hookrightarrow$ read *conjunctively*:
  $\vdash \Gamma$ iff $\bigwedge \Gamma$ is *unsatisfiable*
  $\hookrightarrow \vdash \varphi, \overline{\psi}$ iff $\varphi \to \psi$ is valid
- take finite proof π of $\varphi, \overline{\psi}$
- split every sequent Γ, Δ into $\Gamma \mid \Delta$
- define interpolant θ for every split sequent $\Gamma \mid \Delta$ s.t. $\vdash \Gamma \mid \theta$ and $\vdash \overline{\theta} \mid \Delta$



Maehara's method

Define interpolants:

- define equations between interpolants
- solve system of equations



$$\dfrac{}{\Gamma, \overline{p} \stackrel{\theta_0}{\mid} \Delta, p}\ \text{Ax1}$$

Define equation $\theta_0 = p$

$$\dfrac{\Gamma \stackrel{\theta_1}{\mid} \Delta, \varphi \qquad \Gamma \stackrel{\theta_2}{\mid} \Delta, \psi}{\Gamma \stackrel{\theta_3}{\mid} \Delta, \varphi \vee \psi}\ \vee$$

Define equation $\theta_3 = \theta_1 \vee \theta_2$

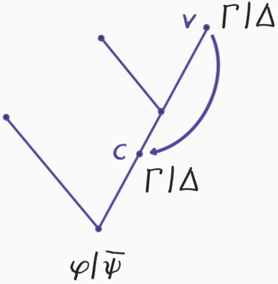
Cyclic proofs

Allow repeat leaves:

- leaves v with ancestor c
- v and c same label

Proofs need to satisfy soundness condition

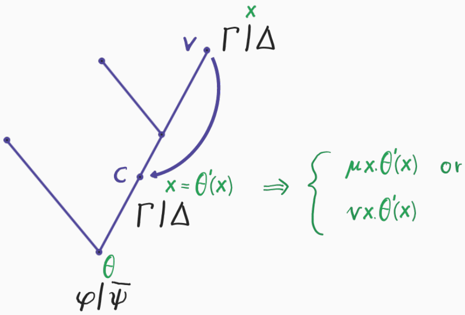
- *global*: on all infinite paths
- *local*: on all repeat paths



Cyclic Maehara's method

Adapt Maehara's method:

- add equation $\theta_v = \theta_c$ for every repeat leaf v with ancestor c
- solve system of fixpoint equations



Cyclic Maehara's method

What is necessary:

1. cut-free (enough: analytic) proof system
2. local soundness condition
3. system of fixpoint equations solvable in logic

Two-way modal μ-calculus

Modal μ-calculus

The modal μ-calculus is generated by the grammar

$$\varphi \ ::= \ p \ | \ \overline{p} \ | \ x \ | \ \varphi \vee \varphi \ | \ \varphi \wedge \varphi \ | \ \langle a \rangle \varphi \ | \ [a]\varphi \ | \ \mu x.\varphi \ | \ \nu x.\varphi$$

$\mu x.\varphi$: least fixpoint

$\nu x.\varphi$: greatest fixpoint

- $\mu x.\varphi$ and $\nu x.\varphi$ are solutions to fixpoint equation $x = \varphi(x)$
- $\mu x.\varphi = \varphi[\mu x.\varphi / x]$ and $\nu x.\varphi = \varphi[\nu x.\varphi / x]$

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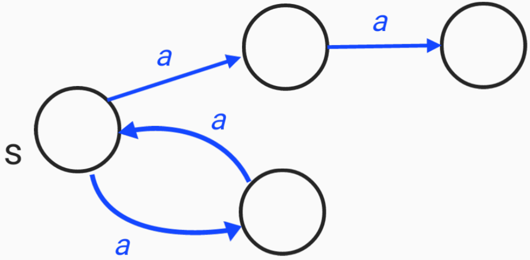
Example:

- $\nu x.\langle a \rangle x$



Properties of modal μ-calculus

- Finite model property
- Satisfiability problem in ExpTime
- Complete axiomatization [Walukiewicz '00]
- Bisimulation-invariant fragment of monadic second-order logic [Janin & Walukiewicz '96]

Properties of modal μ-calculus

- Finite model property
- Satisfiability problem in EXPTIME
- Complete axiomatization [Walukiewicz '00]
- Bisimulation-invariant fragment of monadic second-order logic [Janin & Walukiewicz '96]
- Uniform interpolation property [D'Agostino & Hollenberg '00]
  $\hookrightarrow$ using cyclic proofs [Afshari & Leigh '19]

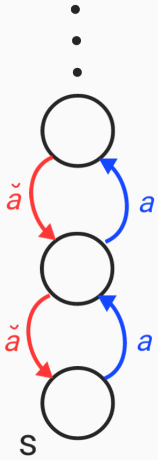
Two-way modal μ-calculus

For every modality a add converse modality $\breve{a}$.

Semantics: $\breve{a}$ interpreted as converse of a

Example:

- $\nu x.(\langle a \rangle x \wedge \mu y\ [\breve{a}]y)$



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- No finite model property
- Tree model property
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  - Automata-theoretic proof [Vardi '98]

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Our contribution:

- Craig interpolation

Proof system NW

An NW-*derivation* is a, possibly infinite, tree defined from the following rules:

$$\text{Ax1: } \frac{}{p, \bar{p}, \Gamma} \qquad \vee\text{: } \frac{\varphi, \psi, \Gamma}{\varphi \vee \psi, \Gamma} \qquad \wedge\text{: } \frac{\varphi, \Gamma \qquad \psi, \Gamma}{\varphi \wedge \psi, \Gamma}$$

$$\langle a \rangle\text{: } \frac{\varphi, \Gamma}{\langle a \rangle \varphi, [a]\Gamma, \Delta} \qquad \mu\text{: } \frac{\varphi[\mu x.\varphi / x], \Gamma}{\mu x.\varphi, \Gamma} \qquad \nu\text{: } \frac{\varphi[\nu x.\varphi / x], \Gamma}{\nu x.\varphi, \Gamma}$$

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- There are infinite branches
- But only finitely many sequents

Example NW-derivation

$$
\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\mu x.\Box x, \nu y.\Diamond y}}{\Box \mu x.\Box x, \Diamond \nu y.\Diamond y}\ \Diamond}{\Box \mu x.\Box x, \nu y.\Diamond y}\ \nu}{\dfrac{\mu x.\Box x, \nu y.\Diamond y}{\mu x.\Box x \vee \Diamond \nu y.\Diamond y}\ \vee}\ \mu
$$

Figure 1: NW-derivation of $\mu x.\Box x \vee \nu y.\Diamond y$.

NW-proofs

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- A trace is a μ-trace if the most important fixpoint formula unfolded infinitely often is a μ-formula.

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Definition

An NW-*proof* is an NW-derivation, where on every infinite branch there is a μ-trace.

Theorem [Niwiński, Walukiewicz '96]

NW $\vdash \Gamma$ iff Γ is unsatisfiable.

Example NW-proof

$$
\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\mu x.\Box x, \nu y.\Diamond y}}{\Box \mu x.\Box x, \Diamond \nu y.\Diamond y}\;\Diamond}{\Box \mu x.\Box x, \nu y.\Diamond y}\;\nu}{\dfrac{\mu x.\Box x, \nu y.\Diamond y}{\mu x.\Box x \vee \Diamond \nu y.\Diamond y}\;\vee}\;\mu
$$

Figure 2: NW-derivation of $\mu x.\Box x \vee \nu y.\Diamond y$.

Proof system for two-way modal μ-calculus

Complications:

- No cut-free sequent system for two-way modal logic
  $\Rightarrow$ add analytic cuts

$$\text{acut: } \frac{\varphi, \Gamma \qquad \overline{\varphi}, \Gamma}{\Gamma} \; \varphi \in \mathsf{Clos}(\Gamma) \qquad\qquad \langle a \rangle\text{: } \frac{\varphi, \Gamma, \langle \breve{a} \rangle \Delta}{\langle a \rangle \varphi, [a]\Gamma, \Delta}$$

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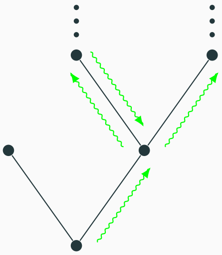
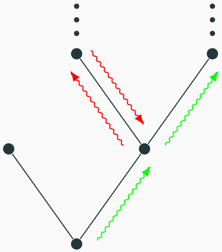
- Traces may go up
  and down in proof tree
  $\Rightarrow$ add trace atoms



Trace atoms

Idea:

- split up traces into upward traces and detour traces
- model detour traces with *trace atoms*: $\varphi \rightsquigarrow_k \psi$
- sequents of form Γ, T



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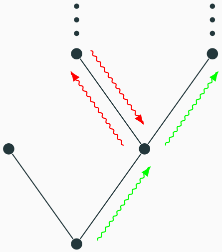

Example rules:

tcut: $$\frac{\varphi \rightsquigarrow_k \psi, \Gamma \qquad \varphi \not\rightsquigarrow_k \psi, \Gamma}{\Gamma}$$

μ: $$\frac{\varphi[\mu x.\varphi/x], \mu x.\varphi \rightsquigarrow_k \varphi[\mu x.\varphi/x], \Gamma}{\mu x.\varphi, \Gamma}$$

Proof system NW_2

- A trace on an infinite branch is an infinite sequence of formulas connected by either
  (i) ancestry, or
  (ii) trace atoms
- A trace is a μ-trace if the most important fixpoint formula unfolded infinitely often is a μ-formula.

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Definition

An NW_2-*proof* is an NW_2-derivation, where on every infinite branch there is a μ-trace.

Theorem

$\mathsf{NW}_2 \vdash \Gamma$ iff Γ is unsatisfiable.

Example NW_2 proof

Let $\varphi := \mu x.[a][\breve{a}]x$

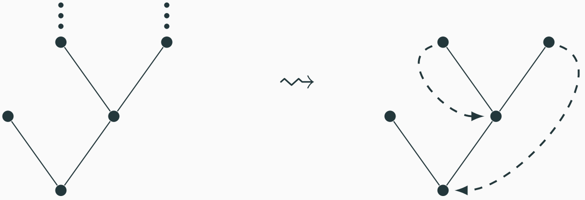
$$\dfrac{\dfrac{\overline{p, [\breve{a}]\varphi, [\breve{a}]\varphi \rightsquigarrow_2 [\breve{a}]\varphi}\ \text{Ax4}}{\langle a\rangle p, [a][\breve{a}]\varphi, \varphi \rightsquigarrow_2 [a][\breve{a}]\varphi}\ \langle a\rangle}{\langle a\rangle p, \mu x.[a][\breve{a}]x}\ \mu$$

Obtaining cyclic proof system

Idea: take finite representations of regular proofs

$\Rightarrow$ global soundness condition:

on every path through the cyclic proof there is a μ-trace



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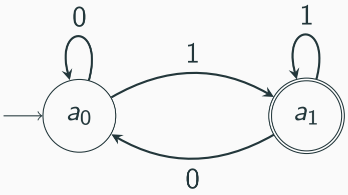
Need better handle on traces: ω-automata

ω-Automata

Büchi automaton:



ω-**Automata**

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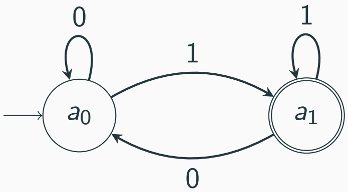
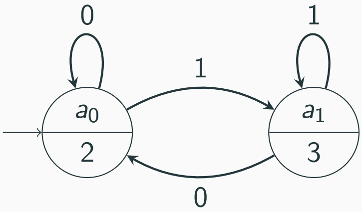


Parity automaton:



ω-Automata

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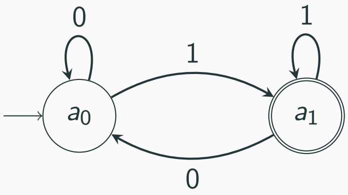
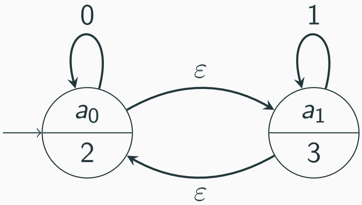


Parity automaton with ε-transitions:



Recap proof system NW_2

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Definition

An NW_2-*proof* is an NW_2-derivation, where on every infinite branch there is a μ-trace.

Theorem

$\mathsf{NW}_2 \vdash \Gamma$ iff Γ is unsatisfiable.

Tracking automaton

Define nondeterministic parity automaton with ε-transitions $\mathbb{A}$ s.t. for all infinite branches α in an NW_2-derivation:

$$\mathbb{A} \text{ accepts } \alpha \;\Leftrightarrow\; \text{there is a } \mu\text{-trace on } \alpha$$

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$$\mathbb{A} \text{ accepts } \alpha \;\Leftrightarrow\; \text{there is a } \mu\text{-trace on } \alpha$$

Idea:

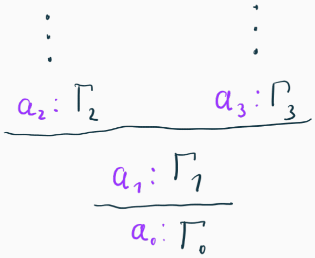
- states are formulas
- transitions given by trace relation
- priorities of fixpoint formulas:
  - μ-formulas even
  - ν-formulas odd
  - more important fixpoint formulas get higher priority

Use tracking automaton

Idea: build automaton into proof system

- Sequents $a : \Gamma$, where a state of $\mathbb{A}$

Need automaton to be deterministic!



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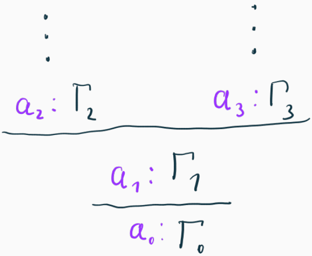
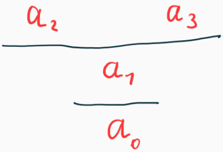


Let $\mathbb{A}^D$ be *deterministic* automaton accepting same language as $\mathbb{A}$

- Sequents $a : \Gamma$, where a : state of $\mathbb{A}^D$

Advantage: Soundness condition based on branches instead of traces



Proof system JS_2

Developed explicit determinization method

$\Rightarrow$ obtain proof system JS_2

- Sequents $\theta : \varphi_1^{\sigma_1}, ..., \varphi_n^{\sigma_n}, T$ where $\theta, \sigma_1, ..., \sigma_n$ list of *names*
- Soundness condition: there is name that 'progresses' infinitely often

Proof system JS_2

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Make proof system cyclic with local soundness condition:

- on all repeat paths there is a name that 'progresses'

Theorem

$JS_2 \vdash \Gamma$ iff Γ is unsatisfiable.

Interpolation

Some technical changes: Transform JS_2 to proof system Circ_2

- Cyclic Maehara method can be applied to Circ_2

Theorem [K,Venema '25]

The two-way modal μ-calculus has Craig interpolation.

Converse PDL

Converse PDL

Converse PDL: PDL with converse modalities

Formulas and *programs* of Converse PDL are given by:

$$\varphi ::= \top \mid \bot \mid p \mid \overline{p} \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \langle\alpha\rangle\varphi \mid [\alpha]\varphi$$
$$\alpha ::= a \mid \alpha;\alpha \mid \alpha \cup \alpha \mid \alpha^* \mid \varphi?$$

Corresponds to fragment of two-way modal μ-calculus

- put syntactic restriction on $\mu x.\varphi$ and $\nu x.\varphi$

Strategy

Cyclic Maehara:

- cyclic proof system with focus-annotations
- define system of fixpoint-equations

  $\Rightarrow$ not solvable inside CPDL

Strategy

Cyclic Maehara:

- cyclic proof system with focus-annotations
- define system of fixpoint-equations

  $\Rightarrow$ not solvable inside CPDL
- translate to different system of fixpoint-equations

  $\Rightarrow$ equivalent and solvable inside CPDL

Theorem [K, Trucco Dalmas, Venema '25]

Converse PDL has Craig interpolation.

Conclusions

- explained Cyclic Maehara method
- proved Craig interpolation property for
  (i) Two-way modal μ-calculus
  (ii) Converse PDL

Conclusions

- explained Cyclic Maehara method
- proved Craig interpolation property for
  (i) Two-way modal μ-calculus
  (ii) Converse PDL
- *Future work:*
  - other logics
  - uniform interpolation

Invitation

Workshop on Fixpoint Logics And Proof Systems
Thursday 12 March, Lab42 in Science Park Amsterdam
More info: https://jkloibhofer.github.io/flaps/

PhD Defense Johannes Kloibhofer
Friday 13 March, 11 am, Aula of the University of Amsterdam

Thank you !

Non-wellfounded proof system NW_2

$$\text{Ax1: } \frac{}{\varphi, \overline{\varphi}, \Gamma} \qquad \text{Ax2: } \frac{}{\bot, \Gamma} \qquad \text{Ax3: } \frac{}{\varphi \rightsquigarrow_k \psi, \varphi \not\rightsquigarrow_k \psi, \Gamma} \qquad \text{Ax4: } \frac{}{\varphi \rightsquigarrow_{2k} \varphi, \Gamma}$$

$$\mathsf{R}_\wedge\text{: } \frac{\varphi, \psi, \varphi \wedge \psi \rightsquigarrow_1 \varphi, \varphi \wedge \psi \rightsquigarrow_1 \psi, \Gamma}{\varphi \wedge \psi, \Gamma} \qquad \mathsf{R}_\vee\text{: } \frac{\varphi, \varphi \vee \psi \rightsquigarrow_1 \varphi, \Gamma \quad \psi, \varphi \vee \psi \rightsquigarrow_1 \psi, \Gamma}{\varphi \vee \psi, \Gamma}$$

$$\mathsf{R}_\eta\text{: } \frac{\varphi[\eta x.\varphi/x], \eta x.\varphi \rightsquigarrow_{\Omega(\eta x.\varphi)} \varphi[\eta x.\varphi/x], \Gamma}{\eta x.\varphi, \Gamma} \qquad \text{trans: } \frac{\varphi \rightsquigarrow_k \psi, \psi \rightsquigarrow_l \chi, \varphi \rightsquigarrow_{\max\{k,l\}} \chi, \Gamma}{\varphi \rightsquigarrow_k \psi, \psi \rightsquigarrow_l \chi, \Gamma} \qquad \text{weak: } \frac{\Gamma}{A, \Gamma}$$

$$\mathsf{R}_{\langle a \rangle}\text{: } \frac{\varphi, \Sigma, \langle \breve{a} \rangle \Gamma, \Gamma^{\langle a \rangle \varphi}}{\langle a \rangle \varphi, [a]\Sigma, \Gamma} \qquad \text{cut: } \frac{\varphi, \Gamma \quad \overline{\varphi}, \Gamma}{\Gamma} \;\; \varphi \in \mathsf{Clos}^\neg(\Gamma) \qquad \text{tcut: } \frac{\varphi \rightsquigarrow_k \psi, \Gamma \quad \varphi \not\rightsquigarrow_k \psi, \Gamma}{\Gamma} \;\; \varphi \in \mathsf{Clos}^\neg(\Gamma)$$

Cyclic proof system JS_2

$$\text{Ax1: } \frac{}{\theta \vdash \varphi^{\sigma}, \overline{\varphi}^{\tau}, \Gamma} \qquad \text{Ax2: } \frac{}{\theta \vdash \bot^{\sigma}, \Gamma} \qquad \text{Ax3: } \frac{}{\theta \vdash \varphi \rightsquigarrow_k \psi, \varphi \not\rightsquigarrow_k \psi, \Gamma} \qquad \text{Ax4: } \frac{}{\theta \vdash \varphi \rightsquigarrow_{2k} \varphi, \Gamma}$$

$$\mathsf{R}_{\wedge}\text{: } \frac{\theta \vdash \varphi^{\sigma}, \psi^{\sigma}, \varphi \wedge \psi \rightsquigarrow_1 \varphi, \varphi \wedge \psi \rightsquigarrow_1 \psi, \Gamma}{\theta \vdash (\varphi \wedge \psi)^{\sigma}, \Gamma} \qquad \mathsf{R}_{\vee}\text{: } \frac{\theta \vdash \varphi^{\sigma}, \varphi \vee \psi \rightsquigarrow_1 \varphi, \Gamma \quad \theta \vdash \psi^{\sigma}, \varphi \vee \psi \rightsquigarrow_1 \psi, \Gamma}{\theta \vdash (\varphi \vee \psi)^{\sigma}, \Gamma}$$

$$\mathsf{R}_{\mu}\text{: } \frac{\theta \cdot \mathsf{x} \vdash \varphi[\mu x.\varphi/x]^{\sigma\lfloor k \cdot \mathsf{x}}, \mu x.\varphi \rightsquigarrow_k \varphi[\mu x.\varphi/x], \Gamma}{\theta \vdash \mu x.\varphi^{\sigma}, \Gamma} \quad k = \Omega(\mu x.\varphi) \text{ and } \mathsf{x} \text{ is a fresh } k\text{-name}$$

$$\mathsf{R}_{\nu}\text{: } \frac{\theta \vdash \varphi[\nu x.\varphi/x]^{\sigma\lfloor k}, \nu x.\varphi \rightsquigarrow_k \varphi[\nu x.\varphi/x], \Gamma}{\theta \vdash \nu x.\varphi^{\sigma}, \Gamma} \quad k = \Omega(\nu x.\varphi) \qquad \mathsf{R}_{\langle \mathsf{a} \rangle}\text{: } \frac{\theta \vdash \varphi^{\sigma}, \Sigma, \langle \breve{a} \rangle \Gamma^{\varepsilon}, \Gamma^{\langle a \rangle \varphi}}{\theta \vdash \langle a \rangle \varphi^{\sigma}, [a]\Sigma, \Gamma}$$

$$\text{trans: } \frac{\theta \vdash \varphi \rightsquigarrow_k \psi, \psi \rightsquigarrow_l \chi, \varphi \rightsquigarrow_{\max\{k,l\}} \chi, \Gamma}{\theta \vdash \varphi \rightsquigarrow_k \psi, \psi \rightsquigarrow_l \chi, \Gamma} \qquad \text{weak: } \frac{\theta \vdash \Gamma}{\theta \vdash A, \Gamma} \qquad \text{exp: } \frac{\theta' \vdash \varphi^{\tau}, \Gamma}{\theta \vdash \varphi^{\sigma}, \Gamma} \quad \theta' \sqsubseteq \theta \text{ and } \tau \sqsubseteq \sigma$$

$$\text{jump}_o\text{: } \frac{\theta \vdash \varphi^{\sigma}, \psi^{\sigma\lfloor 2k+1}, \psi^{\tau}, \varphi \rightsquigarrow_{2k+1} \psi, \Gamma}{\theta \vdash \varphi^{\sigma}, \psi^{\tau}, \varphi \rightsquigarrow_{2k+1} \psi, \Gamma} \qquad \text{jump}_e\text{: } \frac{\theta \cdot \mathsf{x} \vdash \varphi^{\sigma}, \psi^{\sigma\lfloor 2k \cdot \mathsf{x}}, \psi^{\tau}, \varphi \rightsquigarrow_{2k} \psi, \Gamma}{\theta \vdash \varphi^{\sigma}, \psi^{\tau}, \varphi \rightsquigarrow_{2k} \psi, \Gamma} \quad \mathsf{x} \text{ is a fresh } 2k\text{-name}$$

$$\text{cut: } \frac{\theta \vdash \varphi^{\varepsilon}, \Gamma \quad \theta \vdash \overline{\varphi}^{\varepsilon}, \Gamma}{\theta \vdash \Gamma} \quad \varphi \in \mathsf{Clos}^{\neg}(\Gamma) \qquad \text{tcut: } \frac{\theta \vdash \varphi \rightsquigarrow_k \psi, \Gamma \quad \theta \vdash \varphi \not\rightsquigarrow_k \psi, \Gamma}{\theta \vdash \Gamma} \quad \varphi \in \mathsf{Clos}^{\neg}(\Gamma)$$

$$\text{Reset}_{\mathsf{x}}\text{: } \frac{\theta \vdash \varphi_1^{\sigma\mathsf{x}}, \ldots, \varphi_n^{\sigma\mathsf{x}}, \Gamma}{\theta \vdash \varphi_1^{\sigma\mathsf{x}\mathsf{x}_1\tau_1}, \ldots, \varphi_n^{\sigma\mathsf{x}\mathsf{x}_n\tau_n}, \Gamma} \quad \mathsf{x}, \mathsf{x}_1, \ldots, \mathsf{x}_n \text{ are } k\text{-names, } \mathsf{x} \text{ not in } \Gamma \qquad \mathsf{D}_{\mathsf{d}}\text{: } \begin{array}{c} [\theta \vdash \Gamma]^{\mathsf{d}} \\ \vdots \\ \dfrac{\theta \vdash \Gamma}{\theta \vdash \Gamma} \end{array}$$