Interpolation for the two-way modal μ-calculus

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Outline

- Introduce Two-way modal μ-calculus
- What is interpolation?
- Proof strategy: Maehara's method adapted for cyclic proofs
- How to obtain such a proof system

Modal μ-calculus

The modal μ-calculus is generated by the grammar

$$\varphi \ ::= \ p \ | \ \overline{p} \ | \ x \ | \ \varphi \vee \varphi \ | \ \varphi \wedge \varphi \ | \ \langle a \rangle \varphi \ | \ [a]\varphi \ | \ \mu x \ \varphi \ | \ \nu x \ \varphi$$

- $\mu x \ \varphi$: least fixpoint
- $\nu x \ \varphi$: greatest fixpoint

- $\mu x \ \varphi \equiv \varphi[\mu x \ \varphi / x]$
- $\nu x \ \varphi \equiv \varphi[\nu x \ \varphi / x]$

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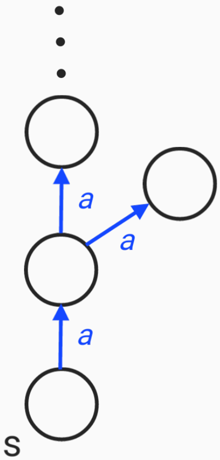
Example:

- $\nu x\ \langle a \rangle x$



Properties of modal μ-calculus

- Originates from [Scott & Bakker '69]
- Modern version introduced by [Kozen '82]
- Key tool in formal study of behaviour of programs
- Fragments of the modal μ-calculus: LTL, CTL, PDL,...

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Properties:

- Finite model property
- Satisfiability problem in EXPTIME
- Complete axiomatization [Walukiewicz '00]
- Bisimulation-invariant fragment of monadic second-order logic [Janin & Walukiewicz '96]
- Uniform interpolation property [D'Agostino & Hollenberg '00]

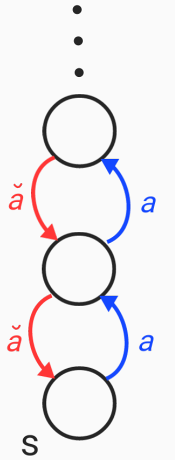
Two-way modal μ-calculus

For every modality a add converse modality $\breve{a}$.

Semantics: $\breve{a}$ interpreted as converse of a

Example:

- $\nu x\ (\langle a\rangle x \wedge \mu y\ [\breve{a}]y)$



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Our contribution:

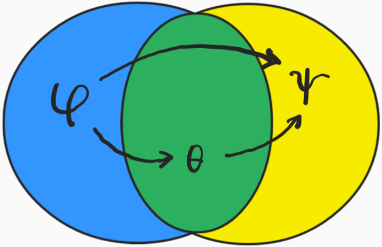
- Craig interpolation

Interpolation

Definition

A logic has Craig interpolation if for any two formulas φ and ψ such that $\varphi \rightarrow \psi$ holds, there is an interpolant θ with

- $\varphi \rightarrow \theta$ and $\theta \rightarrow \psi$
- $\text{Voc}(\theta) \subseteq \text{Voc}(\varphi) \cap \text{Voc}(\psi)$



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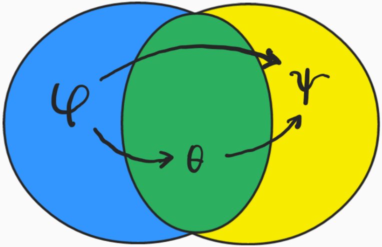
Corrollary: Beth definability

Relevant in

- model checking
- knowledge representation
- ...



Proof methods

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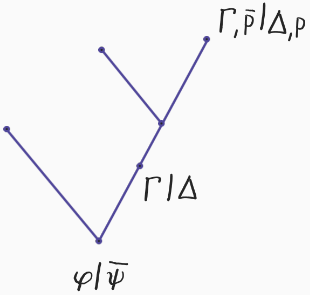
- Split every sequent into $\Gamma \mid \Delta$
- Define interpolant inductively



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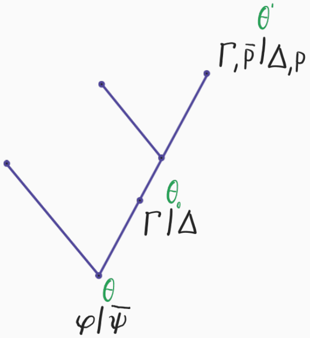


$$\frac{}{\Gamma, \overline{p} \mid \Delta, p}\ \text{Ax1}$$

Define interpolant for $\Gamma, \overline{p} \mid \Delta, p$ as $\theta' := p$

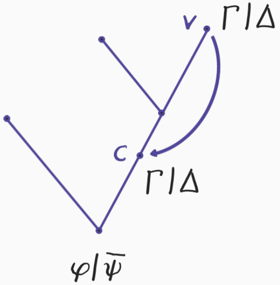
$$\frac{\Gamma \overset{\theta_1}{\mid} \Delta, \varphi \qquad \Gamma \overset{\theta_2}{\mid} \Delta, \psi}{\Gamma \mid \Delta, \varphi \vee \psi}\ \vee$$

Define interpolant for $\Gamma \mid \Delta, \varphi \vee \psi$ as $\theta_0 := \theta_1 \vee \theta_2$

Maehara's method for cyclic proofs

Allow discharged leaves:

- leaves v with ancestor c
- v and c same label
- path from c to v successful



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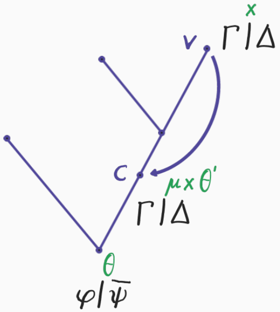
Interpolation idea:

- Assign fresh variable x to v
- Bind x at c with fixpoint



Non-wellfounded proof system NW^2

- Inspired by tableaux games for the modal μ-calculus of [Niwiński & Walukiewicz '96]
- Contains infinite branches
- *Success condition*: Every infinite branch contains infinite trace dominated by μ

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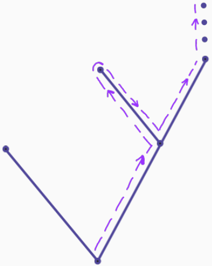
Challenges:

- No cut-free sequent system for two-way modal logic
  $\Rightarrow$ add only analytic cuts
- Traces may go up and down in proof tree
  $\Rightarrow$ add trace atoms



Cyclic proof system JS^2

- Inspired by cyclic proof system of [Jungteerapanich '10] and [Stirling '14]
- Adding annotations to sequents

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Challenges:

- *Completeness*: Use ω-automaton checking success condition on infinite branches in NW^2 proof
  $\Rightarrow$ develop determinization method
- adapt Maehara's method for JS^2

Conclusions

- Introduced two sound and complete proof systems for the two-way modal μ-calculus
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- *Future work:*
    - Uniform interpolation
    - Craig interpolation for Converse PDL

Thank you !

Non-wellfounded proof system NW^2

$$\text{Ax1: } \frac{}{\varphi, \overline{\varphi}, \Gamma} \qquad \text{Ax2: } \frac{}{\bot, \Gamma} \qquad \text{Ax3: } \frac{}{\varphi \rightsquigarrow_k \psi, \varphi \not\rightsquigarrow_k \psi, \Gamma} \qquad \text{Ax4: } \frac{}{\varphi \rightsquigarrow_{2k} \varphi, \Gamma}$$

$$\mathsf{R}_\wedge\text{: } \frac{\varphi, \psi, \varphi \wedge \psi \rightsquigarrow_1 \varphi, \varphi \wedge \psi \rightsquigarrow_1 \psi, \Gamma}{\varphi \wedge \psi, \Gamma} \qquad \mathsf{R}_\vee\text{: } \frac{\varphi, \varphi \vee \psi \rightsquigarrow_1 \varphi, \Gamma \quad \psi, \varphi \vee \psi \rightsquigarrow_1 \psi, \Gamma}{\varphi \vee \psi, \Gamma}$$

$$\mathsf{R}_\eta\text{: } \frac{\varphi[\eta x.\varphi/x], \eta x.\varphi \rightsquigarrow_{\Omega(\eta x.\varphi)} \varphi[\eta x.\varphi/x], \Gamma}{\eta x.\varphi, \Gamma} \qquad \text{trans: } \frac{\varphi \rightsquigarrow_k \psi, \psi \rightsquigarrow_l \chi, \varphi \rightsquigarrow_{\max\{k,l\}} \chi, \Gamma}{\varphi \rightsquigarrow_k \psi, \psi \rightsquigarrow_l \chi, \Gamma} \qquad \text{weak: } \frac{\Gamma}{A, \Gamma}$$

$$\mathsf{R}_{\langle a \rangle}\text{: } \frac{\varphi, \Sigma, \langle \breve{a} \rangle \Gamma, \Gamma^{\langle a \rangle \varphi}}{\langle a \rangle \varphi, [a]\Sigma, \Gamma} \qquad \text{cut: } \frac{\varphi, \Gamma \quad \overline{\varphi}, \Gamma}{\Gamma} \ \varphi \in \mathsf{Clos}^\neg(\Gamma) \qquad \text{tcut: } \frac{\varphi \rightsquigarrow_k \psi, \Gamma \quad \varphi \not\rightsquigarrow_k \psi, \Gamma}{\Gamma} \ \varphi \in \mathsf{Clos}^\neg(\Gamma)$$

Cyclic proof system JS^2

$$\text{Ax1: } \frac{}{\theta \vdash \varphi^\sigma, \overline{\varphi}^\tau, \Gamma} \qquad \text{Ax2: } \frac{}{\theta \vdash \bot^\sigma, \Gamma} \qquad \text{Ax3: } \frac{}{\theta \vdash \varphi \rightsquigarrow_k \psi, \varphi \not\rightsquigarrow_k \psi, \Gamma} \qquad \text{Ax4: } \frac{}{\theta \vdash \varphi \rightsquigarrow_{2k} \varphi, \Gamma}$$

$$\mathsf{R}_\wedge\text{: } \frac{\theta \vdash \varphi^\sigma, \psi^\sigma, \varphi \wedge \psi \rightsquigarrow_1 \varphi, \varphi \wedge \psi \rightsquigarrow_1 \psi, \Gamma}{\theta \vdash (\varphi \wedge \psi)^\sigma, \Gamma} \qquad \mathsf{R}_\vee\text{: } \frac{\theta \vdash \varphi^\sigma, \varphi \vee \psi \rightsquigarrow_1 \varphi, \Gamma \quad \theta \vdash \psi^\sigma, \varphi \vee \psi \rightsquigarrow_1 \psi, \Gamma}{\theta \vdash (\varphi \vee \psi)^\sigma, \Gamma}$$

$$\mathsf{R}_\mu\text{: } \frac{\theta \cdot \mathsf{x} \vdash \varphi[\mu x.\varphi/x]^{\sigma\restriction k \cdot \mathsf{x}}, \mu x.\varphi \rightsquigarrow_k \varphi[\mu x.\varphi/x], \Gamma}{\theta \vdash \mu x.\varphi^\sigma, \Gamma} \quad k = \Omega(\mu x.\varphi) \text{ and } \mathsf{x} \text{ is a fresh } k\text{-name}$$

$$\mathsf{R}_\nu\text{: } \frac{\theta \vdash \varphi[\nu x.\varphi/x]^{\sigma\restriction k}, \nu x.\varphi \rightsquigarrow_k \varphi[\nu x.\varphi/x], \Gamma}{\theta \vdash \nu x.\varphi^\sigma, \Gamma} \quad k = \Omega(\nu x.\varphi) \qquad \mathsf{R}_{\langle \mathsf{a} \rangle}\text{: } \frac{\theta \vdash \varphi^\sigma, \Sigma, \langle \breve{a} \rangle \Gamma^\varepsilon, \Gamma^{\langle a \rangle \varphi}}{\theta \vdash \langle a \rangle \varphi^\sigma, [a]\Sigma, \Gamma}$$

$$\text{trans: } \frac{\theta \vdash \varphi \rightsquigarrow_k \psi, \psi \rightsquigarrow_l \chi, \varphi \rightsquigarrow_{\max\{k,l\}} \chi, \Gamma}{\theta \vdash \varphi \rightsquigarrow_k \psi, \psi \rightsquigarrow_l \chi, \Gamma} \qquad \text{weak: } \frac{\theta \vdash \Gamma}{\theta \vdash A, \Gamma} \qquad \text{exp: } \frac{\theta' \vdash \varphi^\tau, \Gamma}{\theta \vdash \varphi^\sigma, \Gamma} \quad \theta' \sqsubseteq \theta \text{ and } \tau \sqsubseteq \sigma$$

$$\text{jump}_o\text{: } \frac{\theta \vdash \varphi^\sigma, \psi^{\sigma\restriction 2k+1}, \psi^\tau, \varphi \rightsquigarrow_{2k+1} \psi, \Gamma}{\theta \vdash \varphi^\sigma, \psi^\tau, \varphi \rightsquigarrow_{2k+1} \psi, \Gamma} \qquad \text{jump}_e\text{: } \frac{\theta \cdot \mathsf{x} \vdash \varphi^\sigma, \psi^{\sigma\restriction 2k \cdot \mathsf{x}}, \psi^\tau, \varphi \rightsquigarrow_{2k} \psi, \Gamma}{\theta \vdash \varphi^\sigma, \psi^\tau, \varphi \rightsquigarrow_{2k} \psi, \Gamma} \quad \mathsf{x} \text{ is a fresh } 2k\text{-name}$$

$$\text{cut: } \frac{\theta \vdash \varphi^\varepsilon, \Gamma \quad \theta \vdash \overline{\varphi}^\varepsilon, \Gamma}{\theta \vdash \Gamma} \quad \varphi \in \mathsf{Clos}^\neg(\Gamma) \qquad \text{tcut: } \frac{\theta \vdash \varphi \rightsquigarrow_k \psi, \Gamma \quad \theta \vdash \varphi \not\rightsquigarrow_k \psi, \Gamma}{\theta \vdash \Gamma} \quad \varphi \in \mathsf{Clos}^\neg(\Gamma)$$

$$\text{Reset}_\mathsf{x}\text{: } \frac{\theta \vdash \varphi_1^{\sigma\mathsf{x}}, \ldots, \varphi_n^{\sigma\mathsf{x}}, \Gamma}{\theta \vdash \varphi_1^{\sigma\mathsf{x}\mathsf{x}_1\tau_1}, \ldots, \varphi_n^{\sigma\mathsf{x}\mathsf{x}_n\tau_n}, \Gamma} \quad \mathsf{x}, \mathsf{x}_1, \ldots, \mathsf{x}_n \text{ are } k\text{-names, } \mathsf{x} \text{ not in } \Gamma \qquad \begin{array}{c} \lceil \theta \vdash \Gamma \rceil^{\mathsf{d}} \\ \vdots \\ \mathsf{D}_\mathsf{d}\text{: } \dfrac{\theta \vdash \Gamma}{\theta \vdash \Gamma} \end{array}$$