

# Constructing Cyclic Proofs for Interpolation

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- What is **interpolation**?
- How to prove it using **cyclic proofs**
- Case studies:
  - Modal  $\mu$ -calculus
  - Two-way modal  $\mu$ -calculus
  - Converse PDL

# Interpolation

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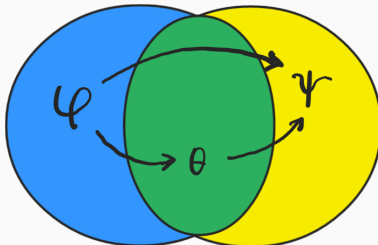
# Interpolation

## Definition

A logic has **Craig interpolation** if for any two formulas  $\varphi$  and  $\psi$  such that  $\varphi \rightarrow \psi$  is valid, there is an **interpolant**  $\theta$  with

- $\varphi \rightarrow \theta$  and  $\theta \rightarrow \psi$  valid
- $\text{Voc}(\theta) \subseteq \text{Voc}(\varphi) \cap \text{Voc}(\psi)$

$\text{Voc}(\varphi)$ : set of proposition letters and modalities in  $\varphi$



# Interpolation

Applications:

- model checking
- knowledge representation
- ...

Proof methods:

- model theory
- algebra
- proof theory

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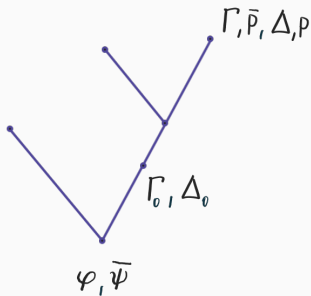
## Proof methods:

- model theory
- algebra
- proof theory: **Maehara's method**

# Maehara's method

Setup:

- start with (cut-free) proof system
  - $\Leftrightarrow$  read *conjunctively*:
    - $\vdash \Gamma$  iff  $\bigwedge \Gamma$  is *unsatisfiable*
    - $\Leftrightarrow \vdash \varphi, \bar{\psi}$  iff  $\varphi \rightarrow \psi$  is valid

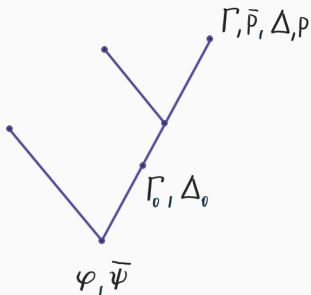


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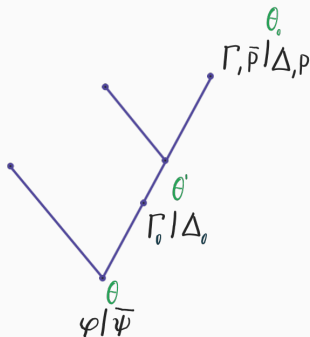
- take **finite** proof  $\pi$  of  $\varphi, \bar{\psi}$
- split every sequent  $\Gamma, \Delta$  into  $\Gamma \mid \Delta$
- define interpolant  $\theta$  for every split sequent  $\Gamma \mid \Delta$  s.t.  
 $\vdash \Gamma \mid \theta$  and  $\vdash \bar{\theta} \mid \Delta$



# Maehara's method

Define interpolants:

- define equations between interpolants
- solve system of equations



$$\frac{}{\Gamma, \bar{p} \mid \Delta, p} \text{Ax1}$$

$$\frac{\Gamma \mid \Delta, \varphi \quad \Gamma \mid \Delta, \psi}{\Gamma \mid \Delta, \varphi \vee \psi} \vee$$

Define equation  $\theta_0 = p$

Define equation  $\theta_3 = \theta_1 \vee \theta_2$

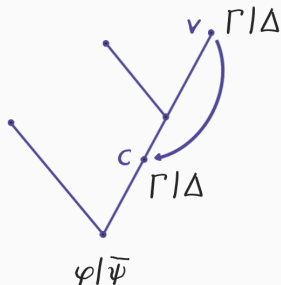
# Cyclic proofs

Allow **repeat leaves**:

- leaves  $v$  with ancestor  $c$
- $v$  and  $c$  same label

Proofs need to satisfy **soundness condition**

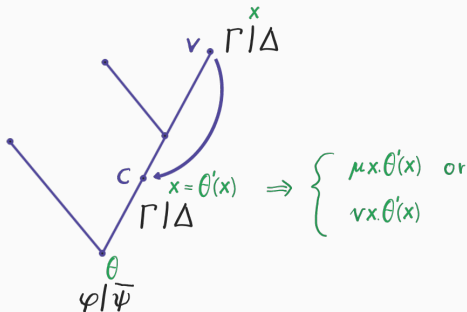
- *global*: on all infinite paths
- *local*: on all repeat paths



# Cyclic Maehara's method

*Adapt Maehara's method:*

- add equation  $\theta_v = \theta_c$  for every repeat leaf  $v$  with ancestor  $c$
- solve system of **fixpoint equations**



What is necessary:

1. cut-free (enough: analytic) cyclic proof system
2. local soundness condition
3. system of fixpoint equations solvable in logic

# Modal $\mu$ -calculus

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# Modal $\mu$ -calculus

The modal  $\mu$ -calculus is generated by the grammar

$$\varphi ::= p \mid \bar{p} \mid x \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \langle a \rangle \varphi \mid [a] \varphi \mid \mu x. \varphi \mid \nu x. \varphi$$

$\mu x. \varphi$ : least fixpoint

$\nu x. \varphi$ : greatest fixpoint

- $\mu x. \varphi$  and  $\nu x. \varphi$  are solutions to fixpoint equation  $x = \varphi(x)$
- $\mu x. \varphi = \varphi[\mu x. \varphi / x]$  and  $\nu x. \varphi = \varphi[\nu x. \varphi / x]$

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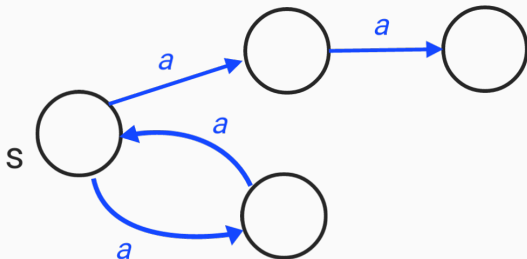
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Example:

- $\nu x. \langle a \rangle x$



## Properties of modal $\mu$ -calculus

- Finite model property
- Satisfiability problem in  $\text{EXPTIME}$
- Complete axiomatization [Walukiewicz '00]
- Bisimulation-invariant fragment of monadic second-order logic [Janin & Walukiewicz '96]

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- Bisimulation-invariant fragment of monadic second-order logic [Janin & Walukiewicz '96]
- **Uniform interpolation property** [D'Agostino & Hollenberg '00]
  - $\leftrightarrow$  using cyclic proofs [Afshari & Leigh '19]

# Proof system NW

An NW-*derivation* is a, possibly infinite, tree defined from the following rules:

$$\text{Ax1: } \frac{}{p, \bar{p}, \Gamma}$$

$$\vee: \frac{\varphi, \psi, \Gamma}{\varphi \vee \psi, \Gamma}$$

$$\wedge: \frac{\varphi, \Gamma \quad \psi, \Gamma}{\varphi \wedge \psi, \Gamma}$$

$$\langle a \rangle: \frac{\varphi, \Gamma}{\langle a \rangle \varphi, [a] \Gamma, \Delta}$$

$$\mu: \frac{\varphi[\mu x. \varphi/x], \Gamma}{\mu x. \varphi, \Gamma}$$

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- There are infinite branches
- But only finitely many sequents

## Example NW-derivation

$$\begin{array}{c} \vdots \\ \frac{\mu x. \Box x, \nu y. \Diamond y}{\Box \mu x. \Box x, \Diamond \nu y. \Diamond y} \quad \Diamond \\ \frac{\Box \mu x. \Box x, \Diamond \nu y. \Diamond y}{\Box \mu x. \Box x, \nu y. \Diamond y} \quad \nu \\ \frac{\Box \mu x. \Box x, \nu y. \Diamond y}{\mu x. \Box x, \nu y. \Diamond y} \quad \mu \\ \frac{\mu x. \Box x, \nu y. \Diamond y}{\mu x. \Box x \vee \Diamond \nu y. \Diamond y} \quad \vee \end{array}$$

**Figure 1:** NW-derivation of  $\mu x. \Box x \vee \nu y. \Diamond y$ .

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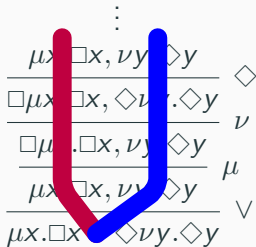
## Definition

An *NW-proof* is an NW-derivation, where on every infinite branch there is a  $\mu$ -trace.

## Theorem [Niwinski, Walukiewicz '96]

$NW \vdash \Gamma$  iff  $\Gamma$  is unsatisfiable.

## Example NW-proof



**Figure 2:** NW-derivation of  $\mu x. \Box x \vee \nu y. \Diamond y$ .

# Obtaining cyclic proof system

Take finite representations of regular proofs



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**Global** soundness condition:

On every infinite path through proof there is a  $\mu$ -trace

What is necessary for cyclic Maehara:

1. cut-free cyclic proof system

## Cyclic Maehara with NW?

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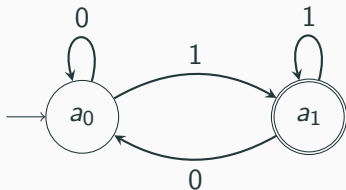
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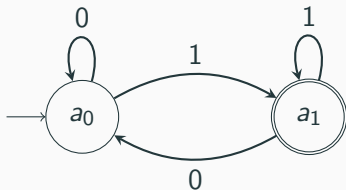
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Need better handle on traces:  $\omega$ -automata

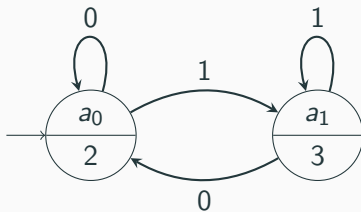
Büchi automaton:



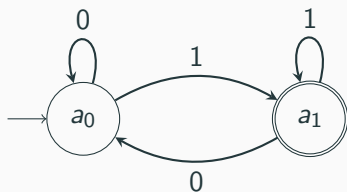
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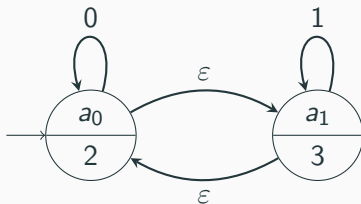
Parity automaton:



Büchi automaton:



Parity automaton with  $\varepsilon$ -transitions:



## Recap proof system NW

- A **trace** on an infinite branch is an infinite sequence of formulas along the branch
- A trace is a  **$\mu$ -trace** if the most important fixpoint formula unfolded infinitely often is a  $\mu$ -formula.

### Definition

An  $NW_2$ -*proof* is an  $NW_2$ -derivation, where on every infinite branch there is a  $\mu$ -trace.

### Theorem

$NW_2 \vdash \Gamma$  iff  $\Gamma$  is unsatisfiable.

# Tracking automaton

Define non-deterministic parity automaton  $\mathbb{A}$  s.t. for all infinite branches  $\beta$  in an NW-derivation:

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Idea:

- states are formulas
- transitions given by trace relation
- priorities of fixpoint formulas:
  - $\mu$ -formulas even
  - $\nu$ -formulas odd
  - more important fixpoint formulas get higher priority

# Use tracking automaton

Idea: build automaton into proof system

- Sequents  $a : \Gamma$ , where  $a$  state of  $\mathbb{A}$

Need automaton to be **deterministic**!

$$\frac{\begin{array}{c} \vdots \\ a_2 : \Gamma_2 \end{array} \quad \begin{array}{c} \vdots \\ a_3 : \Gamma_3 \end{array}}{\frac{a_1 : \Gamma_1}{a_0 : \Gamma_0}}$$

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Need automaton to be **deterministic**!

Let  $\mathbb{A}^D$  be *deterministic* automaton accepting same language as  $\mathbb{A}$

- Sequents  $a : \Gamma$ , where  $a$  state of  $\mathbb{A}^D$

*Advantage*: Soundness condition based on branches instead of traces

$$\frac{\frac{\begin{array}{c} \vdots \\ a_2 : \Gamma_2 \end{array} \quad \begin{array}{c} \vdots \\ a_3 : \Gamma_3 \end{array}}{\quad} \quad a_1 : \Gamma_1}{a_0 : \Gamma_0}$$

$$\frac{a_2 \quad a_3}{a_1} \quad a_0$$

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- first introduced by [Jungteerapanich '10] and [Stirling '14]
- Sequents  $\theta : \varphi_1^{\sigma_1}, \dots, \varphi_n^{\sigma_n}$  where  $\theta, \sigma_1, \dots, \sigma_n$  list of *names*
- Soundness condition: there is name that 'progresses' infinitely often

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Make proof system cyclic with **local** soundness condition:

- on all repeat paths there is a name that 'progresses'

## Theorem

$JS \vdash \Gamma$  iff  $\Gamma$  is unsatisfiable.

What is necessary for cyclic Maehara:

1. cut-free cyclic proof system ✓
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# Interpolation

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Some technical changes: Transform JS to proof system Circ

- Cyclic Maehara method can be applied to Circ

## **Theorem [Afshari,Leigh '19]**

The modal  $\mu$ -calculus has Craig interpolation.

## Two-way modal $\mu$ -calculus

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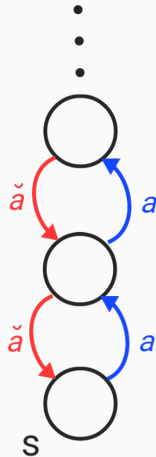
# Two-way modal $\mu$ -calculus

For every modality  $a$  add **converse modality**  $\check{a}$ .

*Semantics:*  $\check{a}$  interpreted as converse of  $a$

Example:

- $\nu x. (\langle a \rangle x \wedge \mu y [\check{a}]y)$



# Properties of two-way modal $\mu$ -calculus

- No finite model property
- Tree model property
- Satisfiability problem in  $\text{EXPTIME}$ 
  - Automata-theoretic proof [Vardi '98]

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Our contribution:

- Craig interpolation

# Proof system for two-way modal $\mu$ -calculus

Extend proof system NW. Complications:

- No cut-free sequent system for two-way modal logic  
 $\Rightarrow$  add **analytic cuts**

$$\text{acut: } \frac{\varphi, \Gamma \quad \bar{\varphi}, \Gamma}{\Gamma} \varphi \in \text{Clos}(\Gamma) \qquad \langle a \rangle: \frac{\varphi, \Gamma, \langle \check{a} \rangle \Delta}{\langle a \rangle \varphi, [a] \Gamma, \Delta}$$

# Proof system for two-way modal $\mu$ -calculus

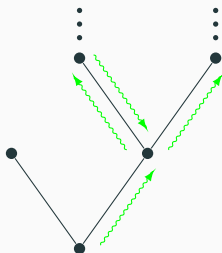
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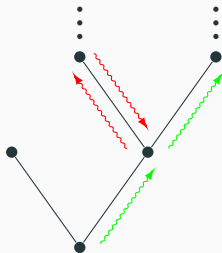
- Traces may go up  
and down in proof tree  
 $\Rightarrow$  add **trace atoms**



# Trace atoms

Idea:

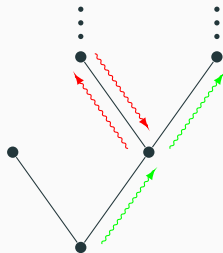
- split up traces into **upward traces** and **detour traces**
- model detour traces with *trace atoms*:  $\varphi \rightsquigarrow_k \psi$
- sequents of form  $\Gamma, T$



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- sequents of form  $\Gamma, T$



Example rules:

$$\text{tcut: } \frac{\varphi \rightsquigarrow_k \psi, \Gamma \quad \varphi \not\rightsquigarrow_k \psi, \Gamma}{\Gamma}$$

$$\mu: \frac{\varphi[\mu x.\varphi/x], \mu x.\varphi \rightsquigarrow_k \varphi[\mu x.\varphi/x], \Gamma}{\mu x.\varphi, \Gamma}$$

- A **trace** on an infinite branch is an infinite sequence of formulas connected by either
  - (i) ancestry, or
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$NW_2 \vdash \Gamma$  iff  $\Gamma$  is unsatisfiable.

## Example NW<sub>2</sub> proof

Let  $\varphi := \mu x. [a][\check{a}]x$

$$\frac{\frac{\frac{}{p, [\check{a}]\varphi, [\check{a}]\varphi \rightsquigarrow_2 [\check{a}]\varphi}{} \text{Ax4}}{\langle a \rangle p, [a][\check{a}]\varphi, \varphi \rightsquigarrow_2 [a][\check{a}]\varphi} \langle a \rangle}{\langle a \rangle p, \mu x. [a][\check{a}]x} \mu$$

## Tracking automaton for $NW_2$

Define non-deterministic  $\varepsilon$ -parity automaton  $\mathbb{A}$  s.t. for all infinite branches  $\beta$  in an  $NW_2$ -derivation:

$\mathbb{A}$  accepts  $\beta \iff$  there is a  $\mu$ -trace on  $\beta$

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Idea:

- $\varepsilon$ -transitions for trace atoms
- normal transitions for ancestor relation

## Proof system JS<sub>2</sub>

Developed determinization method for  $\varepsilon$ -parity automata

⇒ Obtain annotated proof system JS<sub>2</sub>

- Sequents  $\theta : \varphi_1^{\sigma_1}, \dots, \varphi_n^{\sigma_n}, T$  where  $\theta, \sigma_1, \dots, \sigma_n$  list of *names*
- Soundness condition: there is name that 'progresses' infinitely often

## Proof system JS<sub>2</sub>

Developed determinization method for  $\varepsilon$ -parity automata

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- Soundness condition: there is name that ‘progresses’ infinitely often

Make proof system cyclic with **local** soundness condition:

- on all repeat paths there is a name that ‘progresses’

### Theorem

JS<sub>2</sub> ⊢ Γ iff Γ is unsatisfiable.

# Interpolation for the two-way modal $\mu$ -calculus

What is necessary for cyclic Maehara:

1. analytic cyclic proof system ✓
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Technical changes: Transform JS<sub>2</sub> to proof system Circ<sub>2</sub>

- Cyclic Maehara method can be applied to Circ<sub>2</sub>

## **Theorem [K., Venema '25]**

The modal  $\mu$ -calculus has Craig interpolation.

## Converse PDL

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Converse PDL: PDL with converse modalities

*Formulas* and *programs* of Converse PDL are given by:

$$\begin{aligned}\varphi &::= \top \mid \perp \mid p \mid \bar{p} \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \langle \alpha \rangle \varphi \mid [\alpha] \varphi \\ \alpha &::= a \mid \alpha; \alpha \mid \alpha \cup \alpha \mid \alpha^* \mid \varphi?\end{aligned}$$

Corresponds to **fragment** of two-way modal  $\mu$ -calculus

- put syntactic restriction on  $\mu x.\varphi$  and  $\nu x.\varphi$

Developed cyclic proof system with focus-annotations  $\text{CPDL}_f$

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Try cyclic Maehara:

1. analytic cyclic proof system ✓
2. local soundness condition ✓
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# Strategy

Developed cyclic proof system with focus-annotations  $\text{CPDL}_f$

Try cyclic Maehara:

1. analytic cyclic proof system ✓
2. local soundness condition ✓
3. system of fixpoint equations solvable in logic ✗
  - translate to different system of fixpoint-equations  
⇒ equivalent and **solvable** inside CPDL

**Theorem [K, Trucco Dalmas, Venema '25]**

Converse PDL has Craig interpolation.

# Conclusions

- explained Cyclic Maehara method
- proved **Craig interpolation** property for
  - (i) Modal  $\mu$ -calculus
  - (ii) Two-way modal  $\mu$ -calculus
  - (iii) Converse PDL

# Conclusions

- explained Cyclic Maehara method
- proved **Craig interpolation** property for
  - (i) Modal  $\mu$ -calculus
  - (ii) Two-way modal  $\mu$ -calculus
  - (iii) Converse PDL
- *Future work:*
  - other logics
  - uniform interpolation

**Thank you !**

# Non-wellfounded proof system $NW_2$

$$\text{Ax1: } \frac{}{\varphi, \bar{\varphi}, \Gamma}$$

$$\text{Ax2: } \frac{}{\perp, \Gamma}$$

$$\text{Ax3: } \frac{}{\varphi \rightsquigarrow_k \psi, \varphi \not\rightsquigarrow_k \psi, \Gamma}$$

$$\text{Ax4: } \frac{}{\varphi \rightsquigarrow_{2k} \varphi, \Gamma}$$

$$\text{R}_\wedge: \frac{\varphi, \psi, \varphi \wedge \psi \rightsquigarrow_1 \varphi, \varphi \wedge \psi \rightsquigarrow_1 \psi, \Gamma}{\varphi \wedge \psi, \Gamma}$$

$$\text{R}_\vee: \frac{\varphi, \varphi \vee \psi \rightsquigarrow_1 \varphi, \Gamma \quad \psi, \varphi \vee \psi \rightsquigarrow_1 \psi, \Gamma}{\varphi \vee \psi, \Gamma}$$

$$\text{R}_\eta: \frac{\varphi[\eta x. \varphi/x], \eta x. \varphi \rightsquigarrow_{\Omega(\eta x. \varphi)} \varphi[\eta x. \varphi/x], \Gamma}{\eta x. \varphi, \Gamma}$$

$$\text{trans: } \frac{\varphi \rightsquigarrow_k \psi, \psi \rightsquigarrow_l \chi, \varphi \rightsquigarrow_{\max\{k,l\}} \chi, \Gamma}{\varphi \rightsquigarrow_k \psi, \psi \rightsquigarrow_l \chi, \Gamma}$$

$$\text{weak: } \frac{\Gamma}{A, \Gamma}$$

$$\text{R}_{(a)}: \frac{\varphi, \Sigma, \langle \bar{a} \rangle \Gamma, \Gamma^{(a)} \varphi}{\langle a \rangle \varphi, [a] \Sigma, \Gamma}$$

$$\text{cut: } \frac{\varphi, \Gamma \quad \bar{\varphi}, \Gamma}{\Gamma} \quad \varphi \in \text{Clos}^-(\Gamma)$$

$$\text{tcut: } \frac{\varphi \rightsquigarrow_k \psi, \Gamma \quad \varphi \not\rightsquigarrow_k \psi, \Gamma}{\Gamma} \quad \varphi \in \text{Clos}^-(\Gamma)$$

# Cyclic proof system JS<sub>2</sub>

$$\text{Ax1: } \frac{}{\theta \vdash \varphi^\sigma, \overline{\varphi}^\tau, \Gamma}$$

$$\text{Ax2: } \frac{}{\theta \vdash \perp^\sigma, \Gamma}$$

$$\text{Ax3: } \frac{}{\theta \vdash \varphi \rightsquigarrow_k \psi, \varphi \not\rightsquigarrow_k \psi, \Gamma}$$

$$\text{Ax4: } \frac{}{\theta \vdash \varphi \rightsquigarrow_{2k} \varphi, \Gamma}$$

$$\text{R}_\wedge: \frac{\theta \vdash \varphi^\sigma, \psi^\sigma, \varphi \wedge \psi \rightsquigarrow_1 \varphi, \varphi \wedge \psi \rightsquigarrow_1 \psi, \Gamma}{\theta \vdash (\varphi \wedge \psi)^\sigma, \Gamma}$$

$$\text{R}_\vee: \frac{\theta \vdash \varphi^\sigma, \varphi \vee \psi \rightsquigarrow_1 \varphi, \Gamma \quad \theta \vdash \psi^\sigma, \varphi \vee \psi \rightsquigarrow_1 \psi, \Gamma}{\theta \vdash (\varphi \vee \psi)^\sigma, \Gamma}$$

$$\text{R}_\mu: \frac{\theta \cdot x \vdash \varphi[\mu x.\varphi/x]^\sigma |^{k \cdot x}, \mu x.\varphi \rightsquigarrow_k \varphi[\mu x.\varphi/x], \Gamma}{\theta \vdash \mu x.\varphi^\sigma, \Gamma} \quad k = \Omega(\mu x.\varphi) \text{ and } x \text{ is a fresh } k\text{-name}$$

$$\text{R}_\nu: \frac{\theta \vdash \varphi[\nu x.\varphi/x]^\sigma |^k, \nu x.\varphi \rightsquigarrow_k \varphi[\nu x.\varphi/x], \Gamma}{\theta \vdash \nu x.\varphi^\sigma, \Gamma} \quad k = \Omega(\nu x.\varphi)$$

$$\text{R}_{(a)}: \frac{\theta \vdash \varphi^\sigma, \Sigma, \langle \check{a} \rangle \Gamma^\varepsilon, \Gamma^{(a)\varphi}}{\theta \vdash \langle a \rangle \varphi^\sigma, [a]\Sigma, \Gamma}$$

$$\text{trans: } \frac{\theta \vdash \varphi \rightsquigarrow_k \psi, \psi \rightsquigarrow_l \chi, \varphi \rightsquigarrow_{\max\{k,l\}} \chi, \Gamma}{\theta \vdash \varphi \rightsquigarrow_k \psi, \psi \rightsquigarrow_l \chi, \Gamma}$$

$$\text{weak: } \frac{\theta \vdash \Gamma}{\theta \vdash A, \Gamma}$$

$$\text{exp: } \frac{\theta' \vdash \varphi^\tau, \Gamma}{\theta \vdash \varphi^\sigma, \Gamma} \quad \theta' \sqsubseteq \theta \text{ and } \tau \sqsubseteq \sigma$$

$$\text{jump}_o: \frac{\theta \vdash \varphi^\sigma, \psi^\sigma |^{2k+1}, \psi^\tau, \varphi \rightsquigarrow_{2k+1} \psi, \Gamma}{\theta \vdash \varphi^\sigma, \psi^\tau, \varphi \rightsquigarrow_{2k+1} \psi, \Gamma}$$

$$\text{jump}_e: \frac{\theta \cdot x \vdash \varphi^\sigma, \psi^\sigma |^{2k \cdot x}, \psi^\tau, \varphi \rightsquigarrow_{2k} \psi, \Gamma}{\theta \vdash \varphi^\sigma, \psi^\tau, \varphi \rightsquigarrow_{2k} \psi, \Gamma} \quad x \text{ is a fresh } 2k\text{-name}$$

$$\text{cut: } \frac{\theta \vdash \varphi^\varepsilon, \Gamma \quad \theta \vdash \overline{\varphi}^\varepsilon, \Gamma}{\theta \vdash \Gamma} \quad \varphi \in \text{Clos}^-(\Gamma)$$

$$\text{tcut: } \frac{\theta \vdash \varphi \rightsquigarrow_k \psi, \Gamma \quad \theta \vdash \varphi \not\rightsquigarrow_k \psi, \Gamma}{\theta \vdash \Gamma} \quad \varphi \in \text{Clos}^-(\Gamma)$$

$$\text{Reset}_x: \frac{\theta \vdash \varphi_1^{\sigma x}, \dots, \varphi_n^{\sigma x}, \Gamma}{\theta \vdash \varphi_1^{\sigma x_1 \tau_1}, \dots, \varphi_n^{\sigma x_n \tau_n}, \Gamma} \quad x, x_1, \dots, x_n \text{ are } k\text{-names, } x \text{ not in } \Gamma$$

$$\text{D}_d: \frac{[\theta \vdash \Gamma]^d}{\theta \vdash \Gamma}$$