Interpolation for the two-way modal μ-calculus

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Outline

- Introduce Two-way modal μ-calculus
- What is interpolation?
- Proof strategy: Maehara's method adapted for cyclic proofs
- How to obtain such a proof system

Modal μ-calculus

The modal μ-calculus is generated by the grammar

$$\varphi \ ::= \ p \ | \ \overline{p} \ | \ x \ | \ \varphi \vee \varphi \ | \ \varphi \wedge \varphi \ | \ \langle a \rangle \varphi \ | \ [a]\varphi \ | \ \mu x \ \varphi \ | \ \nu x \ \varphi$$

- $\mu x \ \varphi$: least fixpoint
- $\nu x \ \varphi$: greatest fixpoint

- $\mu x \ \varphi \equiv \varphi[\mu x \ \varphi / x]$
- $\nu x \ \varphi \equiv \varphi[\nu x \ \varphi / x]$

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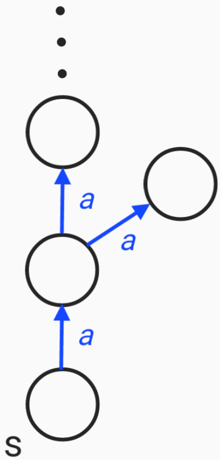
Example:

- $\nu x \ \langle a \rangle x$

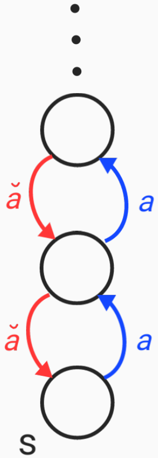


Two-way modal μ-calculus

For every modality a add converse modality $\breve{a}$.
Semantics: $\breve{a}$ interpreted as converse of a

Example:

- $\nu x\ (\langle a \rangle x \wedge \mu y\ [\breve{a}] y)$



Properties of modal μ-calculus

- Finite model property
- Satisfiability problem in ExpTime
- Complete axiomatization [Walukiewicz '00]
- Bisimulation-invariant fragment of monadic second-order logic [Janin & Walukiewicz '96]
- Uniform interpolation property [D'Agostino & Hollenberg '00]

Properties of two-way modal μ-calculus

- No finite model property
- Tree model property
- Satisfiability problem in $\textsc{ExpTime}$
    - Automata-theoretic proof [Vardi '98]

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Our contribution:

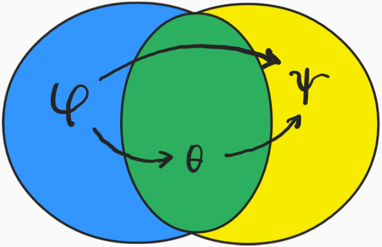
- Craig interpolation

Interpolation

Definition

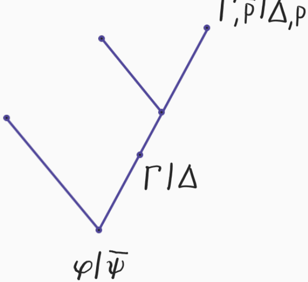
A logic has Craig interpolation if for any two formulas φ and ψ such that $\varphi \to \psi$ holds, there is an interpolant θ with

- $\varphi \to \theta$ and $\theta \to \psi$
- $\mathrm{Voc}(\theta) \subseteq \mathrm{Voc}(\varphi) \cap \mathrm{Voc}(\psi)$



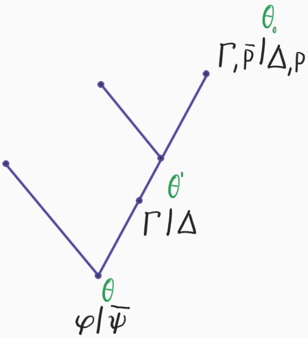
Maehara's method

- Take *finite* proof π of $\varphi, \overline{\psi}$
- Split every sequent into $\Gamma \mid \Delta$



Maehara's method

- Take *finite* proof π of $\varphi, \overline{\psi}$
- Split every sequent into $\Gamma \mid \Delta$
- Define equations between interpolants
- Solve system of equations



$$\dfrac{}{\Gamma, \overline{p} \overset{\theta_0}{\mid} \Delta, p}\ \text{Ax1}$$

Define equation $\theta_0 = p$

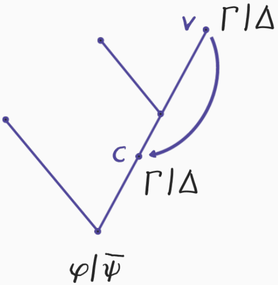
$$\dfrac{\Gamma \overset{\theta_1}{\mid} \Delta, \varphi \qquad \Gamma \overset{\theta_2}{\mid} \Delta, \psi}{\Gamma \overset{\theta_3}{\mid} \Delta, \varphi \vee \psi}\ \vee$$

Define equation $\theta_3 = \theta_1 \vee \theta_2$

Maehara's method for cyclic proofs

Allow discharged leaves:

- leaves v with ancestor c
- v and c same label
- path from c to v successful



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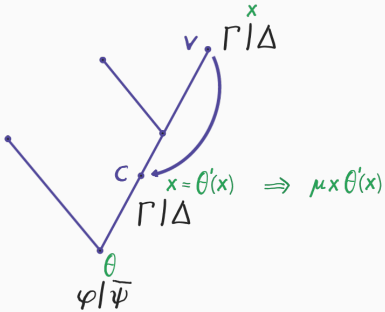

Adapt Maehara's method:

- Add equation $\theta_v = \theta_c = x$ with fresh variable x
- Solve system of fixpoint equations

Non-wellfounded proof system NW^2

- Inspired by tableaux games for the modal μ-calculus of [Niwiński & Walukiewicz '96]
- Contains infinite branches
- *Success condition*: Every infinite branch contains infinite trace dominated by μ

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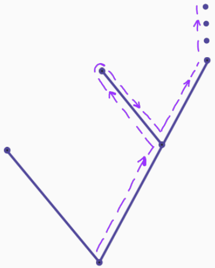
Challenges:

- No cut-free sequent system for two-way modal logic
  $\Rightarrow$ add analytic cuts
- Traces may go up and down in tree model
  $\Rightarrow$ add trace atoms



Cyclic proof system JS^2

- Inspired by cyclic proof system of [Jungteerapanich '10] and [Stirling '14]
- Add annotations to sequents

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Challenges:

- *Completeness*: Use ω-automaton checking success condition on infinite branches in NW^2 proof
  $\Rightarrow$ develop determinization method
- adapt Maehara's method for JS^2

Converse PDL

Converse PDL: PDL with converse modalities

$\Rightarrow$ Fragment of two-way modal μ-calculus

Converse PDL has interpolation [K,Trucco Dalmas & Venema '25]

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Converse PDL has interpolation [K, Trucco Dalmas & Venema '25]

Similar strategy:

- Cyclic proof system with focus-annotations
- Define system of fixpoint-equations

  $\Rightarrow$ not solvable inside CPDL
- Translate to different system of fixpoint-equations

  $\Rightarrow$ equivalent and solvable inside CPDL

Conclusions

- Introduced two sound and complete proof systems for the two-way modal μ-calculus
- Proved Craig interpolation property
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- *Future work:*
  - Uniform interpolation

Thank you !

Non-wellfounded proof system NW^2

$$\text{Ax1: } \frac{}{\varphi, \overline{\varphi}, \Gamma} \qquad \text{Ax2: } \frac{}{\bot, \Gamma} \qquad \text{Ax3: } \frac{}{\varphi \rightsquigarrow_k \psi, \varphi \not\rightsquigarrow_k \psi, \Gamma} \qquad \text{Ax4: } \frac{}{\varphi \rightsquigarrow_{2k} \varphi, \Gamma}$$

$$\mathsf{R}_\wedge\text{: } \frac{\varphi, \psi, \varphi \wedge \psi \rightsquigarrow_1 \varphi, \varphi \wedge \psi \rightsquigarrow_1 \psi, \Gamma}{\varphi \wedge \psi, \Gamma} \qquad \mathsf{R}_\vee\text{: } \frac{\varphi, \varphi \vee \psi \rightsquigarrow_1 \varphi, \Gamma \qquad \psi, \varphi \vee \psi \rightsquigarrow_1 \psi, \Gamma}{\varphi \vee \psi, \Gamma}$$

$$\mathsf{R}_\eta\text{: } \frac{\varphi[\eta x.\varphi/x], \eta x.\varphi \rightsquigarrow_{\Omega(\eta x.\varphi)} \varphi[\eta x.\varphi/x], \Gamma}{\eta x.\varphi, \Gamma} \qquad \text{trans: } \frac{\varphi \rightsquigarrow_k \psi, \psi \rightsquigarrow_l \chi, \varphi \rightsquigarrow_{\max\{k,l\}} \chi, \Gamma}{\varphi \rightsquigarrow_k \psi, \psi \rightsquigarrow_l \chi, \Gamma} \qquad \text{weak: } \frac{\Gamma}{A, \Gamma}$$

$$\mathsf{R}_{\langle a \rangle}\text{: } \frac{\varphi, \Sigma, \langle \breve{a} \rangle \Gamma, \Gamma^{\langle a \rangle \varphi}}{\langle a \rangle \varphi, [a]\Sigma, \Gamma} \qquad \text{cut: } \frac{\varphi, \Gamma \qquad \overline{\varphi}, \Gamma}{\Gamma} \; \varphi \in \mathsf{Clos}^\neg(\Gamma) \qquad \text{tcut: } \frac{\varphi \rightsquigarrow_k \psi, \Gamma \qquad \varphi \not\rightsquigarrow_k \psi, \Gamma}{\Gamma} \; \varphi \in \mathsf{Clos}^\neg(\Gamma)$$

Cyclic proof system JS^2

$$\mathsf{Ax1}: \frac{}{\theta \vdash \varphi^{\sigma}, \overline{\varphi}^{\tau}, \Gamma} \qquad \mathsf{Ax2}: \frac{}{\theta \vdash \bot^{\sigma}, \Gamma} \qquad \mathsf{Ax3}: \frac{}{\theta \vdash \varphi \rightsquigarrow_k \psi, \varphi \not\rightsquigarrow_k \psi, \Gamma} \qquad \mathsf{Ax4}: \frac{}{\theta \vdash \varphi \rightsquigarrow_{2k} \varphi, \Gamma}$$

$$\mathsf{R}_{\wedge}: \frac{\theta \vdash \varphi^{\sigma}, \psi^{\sigma}, \varphi \wedge \psi \rightsquigarrow_1 \varphi, \varphi \wedge \psi \rightsquigarrow_1 \psi, \Gamma}{\theta \vdash (\varphi \wedge \psi)^{\sigma}, \Gamma} \qquad \mathsf{R}_{\vee}: \frac{\theta \vdash \varphi^{\sigma}, \varphi \vee \psi \rightsquigarrow_1 \varphi, \Gamma \quad \theta \vdash \psi^{\sigma}, \varphi \vee \psi \rightsquigarrow_1 \psi, \Gamma}{\theta \vdash (\varphi \vee \psi)^{\sigma}, \Gamma}$$

$$\mathsf{R}_{\mu}: \frac{\theta \cdot \mathsf{x} \vdash \varphi[\mu x.\varphi/x]^{\sigma \upharpoonright k \cdot \mathsf{x}}, \mu x.\varphi \rightsquigarrow_k \varphi[\mu x.\varphi/x], \Gamma}{\theta \vdash \mu x.\varphi^{\sigma}, \Gamma} \quad k = \Omega(\mu x.\varphi) \text{ and } \mathsf{x} \text{ is a fresh } k\text{-name}$$

$$\mathsf{R}_{\nu}: \frac{\theta \vdash \varphi[\nu x.\varphi/x]^{\sigma \upharpoonright k}, \nu x.\varphi \rightsquigarrow_k \varphi[\nu x.\varphi/x], \Gamma}{\theta \vdash \nu x.\varphi^{\sigma}, \Gamma} \quad k = \Omega(\nu x.\varphi) \qquad \mathsf{R}_{\langle \mathsf{a} \rangle}: \frac{\theta \vdash \varphi^{\sigma}, \Sigma, \langle \breve{a} \rangle \Gamma^{\varepsilon}, \Gamma^{\langle a \rangle \varphi}}{\theta \vdash \langle a \rangle \varphi^{\sigma}, [a]\Sigma, \Gamma}$$

$$\mathsf{trans}: \frac{\theta \vdash \varphi \rightsquigarrow_k \psi, \psi \rightsquigarrow_l \chi, \varphi \rightsquigarrow_{\max\{k,l\}} \chi, \Gamma}{\theta \vdash \varphi \rightsquigarrow_k \psi, \psi \rightsquigarrow_l \chi, \Gamma} \qquad \mathsf{weak}: \frac{\theta \vdash \Gamma}{\theta \vdash A, \Gamma} \qquad \mathsf{exp}: \frac{\theta' \vdash \varphi^{\tau}, \Gamma}{\theta \vdash \varphi^{\sigma}, \Gamma} \quad \theta' \sqsubseteq \theta \text{ and } \tau \sqsubseteq \sigma$$

$$\mathsf{jump}_o: \frac{\theta \vdash \varphi^{\sigma}, \psi^{\sigma \upharpoonright 2k+1}, \psi^{\tau}, \varphi \rightsquigarrow_{2k+1} \psi, \Gamma}{\theta \vdash \varphi^{\sigma}, \psi^{\tau}, \varphi \rightsquigarrow_{2k+1} \psi, \Gamma} \qquad \mathsf{jump}_e: \frac{\theta \cdot \mathsf{x} \vdash \varphi^{\sigma}, \psi^{\sigma \upharpoonright 2k \cdot \mathsf{x}}, \psi^{\tau}, \varphi \rightsquigarrow_{2k} \psi, \Gamma}{\theta \vdash \varphi^{\sigma}, \psi^{\tau}, \varphi \rightsquigarrow_{2k} \psi, \Gamma} \quad \mathsf{x} \text{ is a fresh } 2k\text{-name}$$

$$\mathsf{cut}: \frac{\theta \vdash \varphi^{\varepsilon}, \Gamma \quad \theta \vdash \overline{\varphi}^{\varepsilon}, \Gamma}{\theta \vdash \Gamma} \quad \varphi \in \mathsf{Clos}^{\neg}(\Gamma) \qquad \mathsf{tcut}: \frac{\theta \vdash \varphi \rightsquigarrow_k \psi, \Gamma \quad \theta \vdash \varphi \not\rightsquigarrow_k \psi, \Gamma}{\theta \vdash \Gamma} \quad \varphi \in \mathsf{Clos}^{\neg}(\Gamma)$$

$$\mathsf{Reset}_{\mathsf{x}}: \frac{\theta \vdash \varphi_1^{\sigma \mathsf{x}}, \ldots, \varphi_n^{\sigma \mathsf{x}}, \Gamma}{\theta \vdash \varphi_1^{\sigma \mathsf{x} \mathsf{x}_1 \tau_1}, \ldots, \varphi_n^{\sigma \mathsf{x} \mathsf{x}_n \tau_n}, \Gamma} \quad \mathsf{x}, \mathsf{x}_1, \ldots, \mathsf{x}_n \text{ are } k\text{-names, } \mathsf{x} \text{ not in } \Gamma \qquad \begin{array}{c} \lceil \theta \vdash \Gamma \rceil^{\mathsf{d}} \\ \vdots \\ \mathsf{D}_{\mathsf{d}}: \dfrac{\theta \vdash \Gamma}{\theta \vdash \Gamma} \end{array}$$