Cut elimination for Cyclic Proofs: A case study in temporal logic

Bahareh Afshari, Johannes Kloibhofer

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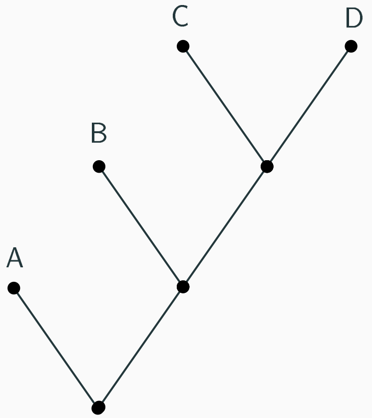
ILLC
University of Amsterdam, Netherlands

Finitary proofs

Strategy:

- ∘ Push cuts upwards
- ∘ Resolve at axioms

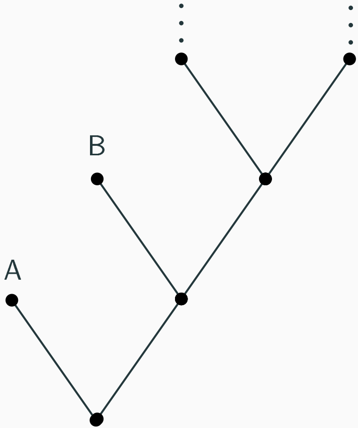


Infinitary proofs

- Contains infinite branches
- Those satisfy *global validity condition*

Strategy:

- Push cuts upwards
- Obtain limit proof
- Check global validity condition

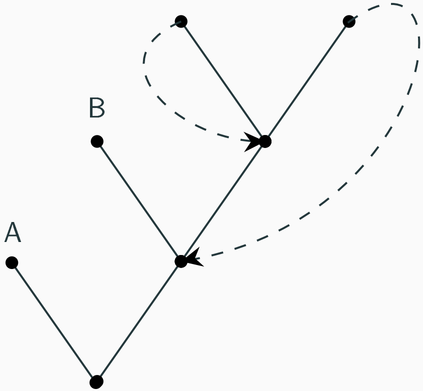


Cyclic proofs

- Contains cycles
- Proof satisfies *global validity condition*

Strategy:

- This Talk!



Modal logic with eventually operator

Formulas of *modal logic with the eventually operator* MLe are defined by

$$\varphi := p \mid \varphi \wedge \varphi \mid \neg\varphi \mid \Diamond\varphi \mid \mathsf{F}\varphi$$

$\mathsf{F}\varphi$ is interpreted as "eventually φ":

- there is a reachable world, where φ holds
- $\mathsf{F}\varphi \equiv \varphi \vee \Diamond\mathsf{F}\varphi$
- in μ-calculus: $\mathsf{F}\varphi \equiv \mu x.(\varphi \vee \Diamond x)$

Proof rules

Ax: $$\frac{}{\varphi \Rightarrow \varphi}$$

$\Diamond$: $$\frac{\varphi \Rightarrow \Gamma}{\Diamond\varphi \Rightarrow \Diamond\Gamma}$$

$\wedge_L$: $$\frac{\varphi, \psi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta}$$

$\wedge_R$: $$\frac{\Gamma \Rightarrow \Delta, \varphi \qquad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi}$$

$\neg_L$: $$\frac{\Gamma \Rightarrow \Delta, \varphi}{\neg\varphi, \Gamma \Rightarrow \Delta}$$

$\neg_R$: $$\frac{\varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\varphi}$$

w_L: $$\frac{\Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}$$

w_R: $$\frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \varphi}$$

F_R: $$\frac{\Gamma \Rightarrow \Delta, \varphi, \Diamond\mathsf{F}\varphi}{\Gamma \Rightarrow \Delta, \mathsf{F}\varphi}$$

F_L: $$\frac{\Diamond\mathsf{F}\varphi, \Gamma \Rightarrow \Delta \qquad \varphi, \Gamma \Rightarrow \Delta}{\mathsf{F}\varphi, \Gamma \Rightarrow \Delta}$$

cut: $$\frac{\Gamma \Rightarrow \Delta, \varphi \qquad \varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

Annotated sequents

Trace condition
A trace τ is *successful* if it is principal in an F_L rule, where it goes to the left.

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We work with annotated sequents:

- Formulas of form $\mathsf{F}\varphi$ or $\Diamond\mathsf{F}\varphi$ *in focus*: $\mathsf{F}\varphi^f$
  or *unfocused*: $\mathsf{F}\varphi^u$
- On every sequent there is at most one formula in antecedent *in focus*

$$\text{Ax: } \frac{}{\varphi^u \Rightarrow \varphi}$$

$$\Diamond\text{: } \frac{\varphi^a \Rightarrow \Gamma}{\Diamond\varphi^a \Rightarrow \Diamond\Gamma}$$

$$\wedge_L\text{: } \frac{\varphi^u, \psi^u, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi^u, \Gamma \Rightarrow \Delta}$$

$$\wedge_R\text{: } \frac{\Gamma \Rightarrow \Delta, \varphi \qquad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi}$$

$$\neg_L\text{: } \frac{\Gamma \Rightarrow \Delta, \varphi}{\neg\varphi^u, \Gamma \Rightarrow \Delta}$$

$$\neg_R\text{: } \frac{\varphi^u, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\varphi}$$

$$\text{w}_L\text{: } \frac{\Gamma \Rightarrow \Delta}{\varphi^u, \Gamma \Rightarrow \Delta}$$

$$\text{w}_R\text{: } \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \varphi}$$

$$\text{F}_R\text{: } \frac{\Gamma \Rightarrow \Delta, \varphi, \Diamond\text{F}\varphi}{\Gamma \Rightarrow \Delta, \text{F}\varphi}$$

$$\text{F}_L\text{: } \frac{\Diamond\text{F}\varphi^a, \Gamma \Rightarrow \Delta \qquad \varphi^u, \Gamma \Rightarrow \Delta}{\text{F}\varphi^a, \Gamma \Rightarrow \Delta}$$

$$\text{cut: } \frac{\Gamma \Rightarrow \Delta, \varphi \qquad \varphi^u, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

$$\text{f: } \frac{\varphi^u, \Gamma \Rightarrow \Delta}{\varphi^f, \Gamma \Rightarrow \Delta}$$

$$\text{u: } \frac{\varphi^f, \Gamma \Rightarrow \Delta}{\varphi^u, \Gamma \Rightarrow \Delta}$$

Successful path

Definition
A $\mathsf{GKe}^{\mathsf{a}}$ *derivation* is defined by the above rules.

Definition
A path τ in a $\mathsf{GKe}^{\mathsf{a}}$ derivation is successful if it holds that

1. Every sequent on τ has a formula in focus and
2. τ passes through an application of F_L, where the principal formula is in focus.

We call a leaf v in a GKe^a derivation π a discharged leaf if there is a proper ancestor $c(v)$ such that

1. v and $c(v)$ are labelled by the same annotated sequent and
2. the path from $c(v)$ to v is *successful*.

Definition
A GKe^a *proof* is a finite GKe^a derivation, where every leaf is either an axiom or a discharged leaf.

Critical case

$$\dfrac{\dfrac{\overset{\pi_0}{\Gamma \Rightarrow \Delta, \varphi, \Diamond F\varphi}}{\Gamma \Rightarrow \Delta, F\varphi}\ F_R \qquad \dfrac{\overset{\pi_1}{\Diamond F\varphi^f, \Gamma \Rightarrow \Delta} \qquad \overset{\pi_2}{\varphi^u, \Gamma \Rightarrow \Delta}}{F\varphi^f, \Gamma \Rightarrow \Delta}\ F_L}{\Gamma \Rightarrow \Delta}\ \text{cut}$$

Critical case

$$\dfrac{\dfrac{\overset{\pi_0}{\Gamma \Rightarrow \Delta, \varphi, \Diamond \mathsf{F}\varphi}}{\Gamma \Rightarrow \Delta, \mathsf{F}\varphi}\ \mathsf{F}_R \qquad \dfrac{\overset{\pi_1}{\Diamond \mathsf{F}\varphi^f, \Gamma \Rightarrow \Delta} \qquad \overset{\pi_2}{\varphi^u, \Gamma \Rightarrow \Delta}}{\mathsf{F}\varphi^f, \Gamma \Rightarrow \Delta}\ \mathsf{F}_L}{\Gamma \Rightarrow \Delta}\ \mathsf{cut}$$

will be transformed to

$$\dfrac{\dfrac{\overset{\pi_0}{\Gamma \Rightarrow \Delta, \varphi, \Diamond \mathsf{F}\varphi} \qquad \overset{\pi_1}{\Diamond \mathsf{F}\varphi^f, \Gamma \Rightarrow \Delta}}{\Gamma \Rightarrow \Delta, \varphi}\ \mathsf{cut} \qquad \overset{\pi_2}{\varphi^u, \Gamma \Rightarrow \Delta}}{\Gamma \Rightarrow \Delta}\ \mathsf{cut}$$

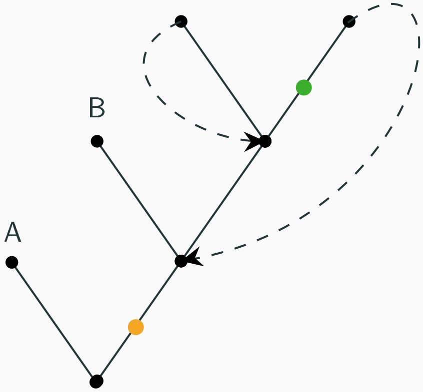
F_L rule is *removed*:

- Might produce unsuccessful paths!

Idea

We call a cut important, if a descendant of the cut-formula is in focus and unimportant otherwise.

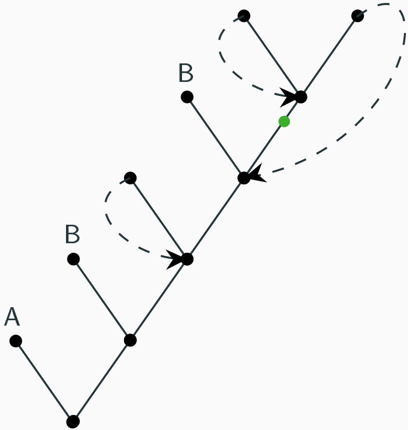
We treat them separately!



Unimportant cuts

Strategy:

- push cut upwards
  - unfold cycles
- reach repeat below cut
- cut unimportant → does not affect formulas in focus
- same formulas in focus → successful repeat



Important cuts

Consider the following important cut:

$$[\Gamma' \Rightarrow \Delta, F\varphi]^{x}$$

$$\vdots \; \pi_v \; \vdots$$

$$\text{v: } \dfrac{\Gamma_v \Rightarrow \Delta_v, \varphi, \Diamond F\varphi}{\Gamma_v \Rightarrow \Delta_v, F\varphi}\; F_R$$

$$\vdots \; \pi_l \; \vdots$$

$$\text{x: } \dfrac{\Gamma' \Rightarrow \Delta, F\varphi}{\Gamma \Rightarrow \Delta, F\varphi}\; u$$

$$[F\varphi^{f}, \Gamma \Rightarrow \Delta]^{y}$$

$$\vdots \; \pi_w^1 \; \vdots \qquad \pi_w^2$$

$$\text{w: } \dfrac{\Diamond F\varphi^{f}, \Gamma_w \Rightarrow \Delta_w \qquad \varphi^{u}, \Gamma_w \Rightarrow \Delta_w}{F\varphi^{f}, \Gamma_w \Rightarrow \Delta_w}\; F_L$$

$$\vdots \; \pi_r \; \vdots$$

$$\text{y: } \dfrac{F\varphi^{f}, \Gamma \Rightarrow \Delta}{F\varphi^{u}, \Gamma \Rightarrow \Delta}\; u$$

$$\dfrac{\Gamma \Rightarrow \Delta, F\varphi \qquad F\varphi^{u}, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\; \text{cut}$$

Important cuts

- Delete desendants of cut formula and 'zip' proof together
- Introduce cut of lower rank

$$
\cfrac{
\text{x: } \cfrac{\begin{array}{c}[\Gamma' \Rightarrow \Delta, F\varphi]^x \\ \vdots \\ \pi_v \\ \vdots \\ \text{v: } \cfrac{\Gamma_v \Rightarrow \Delta_v, \varphi, \Diamond F\varphi}{\Gamma_v \Rightarrow \Delta_v, F\varphi}\ F_R \\ \vdots \\ \pi_l \\ \vdots \\ \Gamma' \Rightarrow \Delta, F\varphi \end{array}}{\Gamma \Rightarrow \Delta, F\varphi}\ u
\qquad
\text{y: } \cfrac{\begin{array}{c}[F\varphi^f, \Gamma \Rightarrow \Delta]^y \\ \vdots \\ \pi_w^1 \\ \vdots \\ \text{w: } \cfrac{\Diamond F\varphi^f, \Gamma_w \Rightarrow \Delta_w \qquad \begin{array}{c}\pi_w^2 \\ \varphi^u, \Gamma_w \Rightarrow \Delta_w\end{array}}{F\varphi^f, \Gamma_w \Rightarrow \Delta_w}\ F_L \\ \vdots \\ \pi_r \\ \vdots \\ F\varphi^f, \Gamma \Rightarrow \Delta \end{array}}{F\varphi^u, \Gamma \Rightarrow \Delta}\ u
}{\Gamma \Rightarrow \Delta}\ \text{cut}
\quad \rightsquigarrow \quad
\begin{array}{c}
[\Gamma' \Rightarrow \Delta]^z \\ \vdots \\ \pi_v, \pi_w^1 \\ \vdots \\
\cfrac{\Gamma_v, \Gamma_w \Rightarrow \Delta_v, \Delta_w, \varphi \qquad \begin{array}{c}\pi_w^2 \\ \varphi^u, \Gamma_w \Rightarrow \Delta_w\end{array}}{\Gamma_v, \Gamma_w \Rightarrow \Delta_v, \Delta_w}\ \text{cut} \\ \vdots \\ \pi_l, \pi_r \\ \vdots \\
\text{z: } \cfrac{\Gamma' \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ u
\end{array}
$$

Conclusion

Summary of method:

- Work with *annotated proof system*
- Treat critical case *separately*
- Usual argumentation for rest
- Produces *cyclic proof* directly

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Extensions to other logics:

- PDL
- Alternation-free μ-calculus
- ...

Thank you !

PDL

Critical case:

$$\dfrac{\begin{array}{c}\pi_0\\ \Gamma \Rightarrow \Delta, \psi, \langle\alpha\rangle\langle\alpha^*\rangle\psi\end{array}}{\Gamma \Rightarrow \Delta, \langle\alpha^*\rangle\psi}\ \langle*\rangle_R \qquad \dfrac{\begin{array}{cc}\pi_1 & \pi_2\\ \langle\alpha\rangle\langle\alpha^*\rangle\psi^f, \Gamma \Rightarrow \Delta & \psi^f, \Gamma \Rightarrow \Delta\end{array}}{\langle\alpha^*\rangle\psi^f, \Gamma \Rightarrow \Delta}\ \langle*\rangle_L$$

- Focus branches
- Introduce multi-cut:

$$\dfrac{\Gamma \Rightarrow \Delta, \psi_1, ..., \psi_n \qquad \psi_1^f, \Gamma \Rightarrow \Delta \qquad \cdots \qquad \psi_n^f, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \text{multicut}$$